Countable Fréchet Groups

Ulises Ariet Ramos García
(joint work with Michael Hrušák)

IMUNAM-Morelia
Universidad Nacional Autónoma de México
ariet@matmor.unam.mx

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Contents

The beginnings

Theorem (Birkhoff-Kakutani, 1936)

Every first countable group is metrizable.

Definition

A topological space X is *Fréchet-Urysohn* (or just Fréchet) if whenever a point $x \in X$ is in the closure of a set A, there is a sequence of elements of A converging to x.

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Every first countable space is Fréchet.

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Example

Let $\mathbb{G} = \{f \in 2^{\omega_1} : |\text{supp}(f)| \leqslant \omega\}$. Then $\mathbb{G}$ is a countably compact Fréchet topological group that is not first countable.

Remark

- *The group $\mathbb{G}$ in the previous example is not countable. In fact, all countable subsets of $\mathbb{G}$ are metrizable (and thus first countable).*
- *If a topological group has a first countable dense subspace then such group is also first countable.*

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Is there a countable (separable) Fréchet topological group which is non metrizable?

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The Boolean case idealized

Definition

Let $\mathcal{I}$ be an (free) ideal on ω. Then

$$\mathcal{I}^{<\omega} = \{J \subseteq [\omega]^{<\omega} : (\exists I \in \mathcal{I})(\forall a \in J)(I \cap a \neq \emptyset)\}$$

is an ideal on $[\omega]^{<\omega}$. Let the dual filter of $\mathcal{I}^{<\omega}$ be a neighbourhood base at $\emptyset$, then use it to give a group topology $\tau_{\mathcal{I}}$ on $([\omega]^{<\omega}, \Delta)$.

Definition

Given a space X and a point $x \in X$ let

- $\mathcal{I}_x = \{I \subseteq X : x \notin \overline{I}\}$ and
- $\mathcal{I}_x^{\perp} = \{J \subseteq X : (\forall I \in \mathcal{I}_x)(|I \cap J| < \omega)\}$, *i.e,* the set of all converging sequences to x.

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The Fréchet property idealized

Remark

- X *is Fréchet at* x *if and only if* $\mathcal{I}_x = \mathcal{I}_x^{\perp\perp}$.
- X *is first countable if and only if* $cof(\mathcal{I}_x) = \omega$.

Definition

Call an ideal $\mathcal{I}$ Fréchet if $\mathcal{I} = \mathcal{I}^{\perp\perp}$.

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Proposition

- *$\tau_{\mathcal{I}}$ is Fréchet if and only if $\mathcal{I}^{<\omega}$ is Fréchet.*
- *$\mathrm{cof}(\mathcal{I}^{<\omega}) = \mathrm{cof}(\mathcal{I})$, so $\tau_{\mathcal{I}}$ is first countable if and only if $\mathrm{cof}(\mathcal{I}) = \omega$.*
- *$\mathrm{cof}(\mathcal{J}) < \mathfrak{p}$ implies $\mathcal{J}$ is Fréchet, for any ideal $\mathcal{J}$.*

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- *cof*$(\mathcal{I}^{<\omega}) =$ *cof*$(\mathcal{I})$*, so* $\tau_{\mathcal{I}}$ *is first countable if and only if* *cof*$(\mathcal{I}) = \omega$*.*
- *cof*$(\mathcal{J}) < \mathfrak{p}$ *implies* $\mathcal{J}$ *is Fréchet, for any ideal* $\mathcal{J}$*.*

Consistent examples

Examples

For each one of the following assumptions, there is an example of the kind $([\omega]^{<\omega}, \tau_{\mathcal{I}})$ to Malykhin's question:

- $\mathfrak{p} > \omega_1$.
- (Nyikos, 1992) $\mathfrak{p} = \mathfrak{b}$.

Definition

$$\mathfrak{b} = \min\{|B| : B \text{ is an unbounded subset of } {}^{\omega}\omega\},$$

$$\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a subfamily of } [\omega]^{\omega} \text{ with the } \textit{sfip}\text{, which has no infinite pseudo-intersection}\}.$$

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Partial negative solution

Theorem (Todorčević-Uzcátegui, 2005)

Every countable Fréchet topological group whose topology is analytic is metrizable.

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γ-sets

Definition

Let $\mathcal{U}$ be an open cover of a space X. Then:

- $\mathcal{U}$ is an *ω-cover* if for every finite set $F \subseteq X$ there is a $U \in \mathcal{U}$ such that $F \subseteq U$.
- $\mathcal{U}$ is a *γ-cover* if every $x \in X$ is cointained in all but finitely many elements of $\mathcal{U}$.

A space X is a γ-space if every ω-cover of X contains a γ-subcover. A γ-space which is separable metric is called γ-set. In the same way we define $\text{non}(\gamma\text{-set}) = \min\{|X| : X \text{ is not a } \gamma\text{-set}\}$.

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Theorem (Gerlits-Nagy)

non(γ-set)$= \mathfrak{p}$ so every separable metric space of size $< \mathfrak{p}$ is a γ-set.

Definition

A set $X \subseteq \mathbb{R}$ has *strong measure zero* (SMZ) if for every sequence of positive reals $\langle \varepsilon_n : n \in \omega \rangle$ there exists a sequence of intervals $\langle I_n : n \in \omega \rangle$ such that $\operatorname{diam}(I_n) \leqslant \varepsilon_n$ for $n \in \omega$ and $X \subseteq \bigcup_{n \in \omega} I_n$.

Theorem (Gerlits-Nagy/Laver)

Every γ-set is SMZ so it is consistent with **ZFC** *that every γ-set is countable.*

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γ-sets and $C_p(X)$ Fréchet

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Corollary (Gerlits-Nagy, 1982)

The existence of $C_p(X)$ which is separable Fréchet non-metrizable is equivalent to the existence of an uncountable γ-set.

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Given an abelian topological group $\mathbb{G}$ its (dual) *group of characters* is

$$\mathbb{G}^* = \{x \colon \mathbb{G} \to \mathbb{T} \colon x \text{ is a continuous homomorphism}\}.$$

with the *compact-open* topology.

Theorem (Pontryagin)

If $\mathbb{G}$ is abelian locally compact then so is $\mathbb{G}^*$ and, morover, $\mathbb{G}^{**}$ is naturally isomorphic to $\mathbb{G}$.

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$\mathbb{G}^*$ is compact if and only if $\mathbb{G}$ is discrete.

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$\mathbb{G}^*$ is compact if and only if $\mathbb{G}$ is discrete.

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Given an abelian topological group $\mathbb{G}$ its (dual) *group of characters* is

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Pre-compact topologies on abelian groups

Definition

A topological group $\mathbb{G}$ is *precompact* (or equivalently totally bounded) if it is a dense subgroup of a compact group (eq. if finitely many translates of very nbhd of *id* cover $\mathbb{G}$).

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Let $\mathbb{G}$ be an abelian group (discrete) and $X \subseteq G^*$. We say that X *separates points* of $\mathbb{G}$ if for every $id \neq g \in \mathbb{G}$ there is an $x \in X$ such that $x(g) \neq 0$.

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Given $\mathbb{G}$ an abelian group and $X \subset \mathbb{G}^*$ that separates points of $\mathbb{G}$ let τ_X be the weakest topology on $\mathbb{G}$ which makes all $x \in X$ continuous.

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$(\mathbb{G}, \tau_X)$ is precompact, morover, every precompact group topology on $\mathbb{G}$ is of the form τ_X.

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$\gamma_{\mathbb{G}}$-sets

Definition

Given a countable abelian group $\mathbb{G}$, $g \in G$ and $m > 0$ let

$$U_g^m = \{x \in \mathbb{G}^* : d(x(g), 0) < \frac{1}{m}\}$$

and given $A \subseteq \mathbb{G}$ let

$$\mathcal{U}_A^m = \{U_g^m : g \in A\}.$$

A set $X \subseteq \mathbb{G}^*$ is $\gamma_{\mathbb{G}}$-*set*, if for every infinite $A \subseteq \mathbb{G}$ if $\mathcal{U}_A^m$ is an ω-cover of X for every $m > 0$, then there is an infinite $B \subseteq A$ such that $\mathcal{U}_B^m$ is a γ-cover of X for every $m > 0$.

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Theorem (Hrušák-R.)

Let $\mathbb{G}$ be an countable abelian group and $X \subseteq \mathbb{G}^*$ separates points of $\mathbb{G}$. Then, $(\mathbb{G}, \tau_X)$ is Fréchet if and only if X is a $\gamma_{\mathbb{G}}$-set.

Remark

If $X \subseteq \mathbb{G}^*$ is a γ-set then X is a $\gamma_{\mathbb{G}}$-set.

Corollary

The existence of a non-metrizable precompact Fréchet group topology on a countable abelian group is equivalent to the existence of an uncountable $\gamma_{\mathbb{G}}$-set.

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$\gamma_{\mathbb{G}}$-sets and **ZFC**

Conjecture

It is consistent with **ZFC** *that every $\gamma_{\mathbb{G}}$-set is countable, or equivalently, it is consistent with* **ZFC** *that every countable abelian precompact Fréchet group is metrizable.*

Questions

- Is there a non-metrizable countable Fréchet group?
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