# Finite chain condition and packing completeness for ideals on countable groups

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## Definition

A family $\mathcal{I}$ of subsets of a group $G$ is *ideal* if

- $G \notin \mathcal{I}$;
- $\mathcal{I}$ is closed under taking subsets;
- $\mathcal{I}$ is closed under finite unions.

Such an ideal $\mathcal{I}$ is called *invariant* if

$$\forall A \in \mathcal{I}\ \forall x \in G\ \ x + A \in \mathcal{I}.$$

**Trivial examples:** $\mathcal{I} = \{\emptyset\}$, $\mathcal{I} = [G]^{<\omega}$.
**Nontrivial Examples:** Ask Jana Flašková.

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# Classical examples of ideals on $\mathbb{R}$:

- $\mathcal{N}$ the ideal of Lebesgue null sets;
- $\mathcal{UN}$ the ideal of universally null sets;
- $\mathcal{M}$ the ideal of meager subsets;
- $\mathcal{UM}$ the ideal of universally meager subsets;
- $\mathcal{US}$ the ideal of universally small subsets.

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The answer is easy for the first two ideals:

Just take any Banach (=shift-invariant finitely additive probability) measure $\mu$ on $G$ and consider the ideal:

- $\mathcal{N}_\mu$ of null subsets of $G$ with respect to the measure $\mu$.

Such ideals are important because of

### Theorem

*Each countably generated invariant ideal $\mathcal{I}$ on a countable abelian group $G$ lies in the ideal $\mathcal{N}_\mu$ for a suitable Banach measure $\mu$.*

The intersection of all null ideals gives the ideal

- $\mathcal{UN} = \bigcap_\mu \mathcal{N}_\mu$ of universally null subsets of $G$.

So we get the inclusion:

$$\mathcal{UN} = \bigcap_\mu \mathcal{N}_\mu \subset \bigcup_\mu \mathcal{N}_\mu.$$

Note that the latter union is not at ideal in $G$ and coincides with the union of all invariant ideals on $G$!

**Question:** What about the ideal $\mathcal{M}$ of meager sets? What can be understood under “nowhere dense” subsets of $G$ (for example, in case $G = \mathbb{Z}$)?

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# Large subsets of groups

A subset $A \subset G$ is *large* if it belongs to no invariant ideal on $G$. This happens if and only if $F + A = G$ for some finite subset $F \subset G$.

**Example:** *Any subset with non-empty interior in a totally bounded topological group G is large.*

Theorem

*A subset $A \subset G$ of a countable abelian group G is large if and only if $\mu(A) > 0$ for every invariant measure $\mu$ on G.*

So, the union $\bigcup_\mu \mathcal{N}_\mu$ equals the union of all ideals on $G$ and consists of all non-large subsets.

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By the way, the following intriguing problem concerning large sets is still open:

### Problem (Ellis)

*Is it true that for each large subset* $A \subset \mathbb{Z}$ *the difference* $A - A$ *is a neighborhood of zero in some totally bounded group topology on* $\mathbb{Z}$.

# Small subsets in groups

**Definition**

A subset $A$ of a group $G$ is *small* if for every large set $L \subset G$ the difference $L \setminus A$ is large.

**Theorem**

*For a subset $A$ of a countable abelian group $G$ TFAE:*

1. *$A$ is small;*
2. *for every finite $F \subset G$ the set $G \setminus (F + A)$ is large;*
3. *$A$ is nowhere dense in some (Hausdorff) totally bounded invariant topology on $G$.*

An invariant topology on $G$ is *totally bounded* if each open non-empty subset of $G$ is large.

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An invariant topology on $G$ is *totally bounded* if each open non-empty subset of $G$ is large.

Thus: small sets are exactly nowhere dense subsets in suitable totally bounded topologies.

It follows from the defintion that the family $\mathcal{S}$ of small subsets of a group is an invariant ideal. This ideal relates to the other ideals as follows:

$$\mathcal{UN} \subset \mathcal{S} \subset \bigcup_{\mu} \mathcal{N}_{\mu}.$$

**Question:** What can be understood by universally small subset?

**Hint:** We need a counterpart of the countable chain condition for ideals in countable groups.

# Packing index

Given a subset $A \subset G$ consider the cardinal

$$\mathrm{pack}(A) = \sup\{|B| : B \subset G \quad \{b + A\}_{b \in B} \text{ is disjoint}\}$$

called *the packing index* of $A$.

**Example:** $\mathrm{pack}(2\mathbb{Z}) = 2$.

# CH for packing indexes

Problem (Omiljanowski)

*Is it true that the packing index* $\mathrm{pack}(A)$ *of a Borel subset of* $\mathbb{R}$ *is either at most countable or else equal to* $\mathfrak{c}$*.*
*(This is true if A is* $\sigma$*-compact.)*

# $\mathcal{I}$-packing index

Let $\mathcal{I}$ is an ideal of subsets of a group.

We define a family $\mathcal{A}$ of subsets of $G$ to be $\mathcal{I}$-*disjoint* if $A \cap A' \in \mathcal{I}$ for any two distinct sets $A, A' \in \mathcal{A}$.

If $\mathcal{I} = \{\emptyset\}$ (resp. $\mathcal{I} = [G]^{<\omega}$), then $\mathcal{I}$-disjoint is the same as (almost) disjoint in the usual sense.

Introducing an ideal parameter in the definition of a packing index, we obtain the notion of the $\mathcal{I}$-*packing index*

$$\mathcal{I}\text{-pack}(A) = \sup\{|B| : B \subset G \ \ \{b + A\}_{b \in B} \text{ is } \mathcal{I}\text{-disjoint}\}.$$

# The packing completeness of ideals

**Definition**

An ideal $\mathcal{I}$ on $G$ is $\mathrm{pack}$-*complete* if each subset $A \subset G$ with $\mathcal{I}\text{-}\mathrm{pack}(A) \geq \aleph_0$ belongs to $\mathcal{I}$.

So, the packing completeness can be thought as a countepart of ccc-property for ideals on countable groups.

# Examples of packing complete ideals:

The following ideals are packing complete:

- $\mathcal{N}_\mu$ for every invariant measure $\mu$ on *G*;
- $\mathcal{UN} = \bigcap_\mu \mathcal{N}_\mu$;
- $\mathcal{S}$, the ideal of small subsets of a countable abelian group *G*.

# The packing completion of an ideal

Theorem

*For every ideal $\mathcal{I}$ on a countable abelian group G the intersection $\tilde{\mathcal{I}}$ of all packing complete ideals that contain $\mathcal{I}$ is a well-defined packing complete ideal called the* packing completion *of $\mathcal{I}$. It is equal to the union*

$$\tilde{\mathcal{I}} = \bigcup_{\alpha<\omega_1} \mathcal{I}_\alpha$$

*where $\mathcal{I}_0 = \mathcal{I}$ and $\mathcal{I}_\alpha$ is the ideal generated by all subsets with infinite $\mathcal{I}_{<\alpha}$-packing index.*

The packing completion $\mathcal{US}$ of the empty ideal $\mathcal{I} = \{\emptyset\}$ is the smallest packing complete ideal. So, we get the chain of packing complete ideals:

$$\mathcal{US} \subset \mathcal{UN} \subset \mathcal{S} \subset \bigcup_{\mu} \mathcal{N}_{\mu}.$$

The last two inclusions cannot be reversed.

## Problem

1. *Is $\mathcal{US} \neq \mathcal{UN}$?*
2. *Find a combinatorial characterization of subsets belonging to the ideal $\mathcal{US}$.*
3. *What is the descriptive complexity of the ideals $\mathcal{US}$ and $\mathcal{UN}$?*

Thank you!