Finite chain condition and packing completeness for ideals on countable groups

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Definition
A family $\mathcal{I}$ of subsets of a group $G$ is *ideal* if
- $G \notin \mathcal{I}$;
- $\mathcal{I}$ is closed under taking subsets;
- $\mathcal{I}$ is closed under finite unions.

Such an ideal $\mathcal{I}$ is called *invariant* if

$$\forall A \in \mathcal{I} \forall x \in G \ x + A \in \mathcal{I}.$$ 

Trivial examples: $\mathcal{I} = \{\emptyset\}$, $\mathcal{I} = [G]^{<\omega}$.

Nontrivial Examples: Ask Jana Flašková.
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Classical examples of ideals on $\mathbb{R}$:

- $\mathcal{N}$ the ideal of Lebesgue null sets;
- $\mathcal{UN}$ the ideal of universally null sets;
- $\mathcal{M}$ the ideal of meager subsets;
- $\mathcal{UM}$ the ideal of universally meager subsets;
- $\mathcal{US}$ the ideal of universally small subsets.

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The answer is easy for the first two ideals:
Just take any Banach (=shift-invariant finitely additive probability) measure \( \mu \) on \( G \) and consider the ideal:

\[ N_\mu \]

of null subsets of \( G \) with respect to the measure \( \mu \).
Such ideals are important because of

**Theorem**

*Each countably generated invariant ideal \( \mathcal{I} \) on a countable abelian group \( G \) lies in the ideal \( N_\mu \) for a suitable Banach measure \( \mu \).*
The intersection of all null ideals gives the ideal

\[ UN = \bigcap_{\mu} N_{\mu} \]

of universally null subsets of \( G \).

So we get the inclusion:

\[ UN = \bigcap_{\mu} N_{\mu} \subseteq \bigcup_{\mu} N_{\mu}. \]

Note that the latter union is not an ideal in \( G \) and coincides with the union of all invariant ideals on \( G \)!

Question: What about the ideal \( M \) of meager sets? What can be understood under “nowhere dense” subsets of \( G \) (for example, in case \( G = \mathbb{Z} \))?
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**Question:** What about the ideal \( \mathcal{M} \) of meager sets? What can be understood under “nowhere dense” subsets of \( G \) (for example, in case \( G = \mathbb{Z} \))?
Large subsets of groups

A subset $A \subset G$ is *large* if it belongs to no invariant ideal on $G$. This happens if and only if $F + A = G$ for some finite subset $F \subset G$.

**Example:** *Any subset with non-empty interior in a totally bounded topological group $G$ is large.*

**Theorem**

A subset $A \subset G$ of a countable abelian group $G$ is large if and only if $\mu(A) > 0$ for every invariant measure $\mu$ on $G$.

So, the union $\bigcup_{\mu} \mathcal{N}_\mu$ equals the union of all ideals on $G$ and consists of all non-large subsets.
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By the way, the following intriguing problem concerning large sets is still open:

**Problem (Ellis)**

*Is it true that for each large subset $A \subset \mathbb{Z}$ the difference $A - A$ is a neighborhood of zero in some totally bounded group topology on $\mathbb{Z}$.*
Definition
A subset $A$ of a group $G$ is small if for every large set $L \subset G$ the difference $L \setminus A$ is large.

Theorem
For a subset $A$ of a countable abelian group $G$ TFAE:
1. $A$ is small;
2. for every finite $F \subset G$ the set $G \setminus (F + A)$ is large;
3. $A$ is nowhere dense in some (Hausdorff) totally bounded invariant topology on $G$.

An invariant topology on $G$ is totally bounded if each open non-empty subset of $G$ is large.
Small subsets in groups

**Definition**
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1. $A$ is small;
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An invariant topology on $G$ is *totally bounded* if each open non-empty subset of $G$ is large.
Thus: small sets are exactly nowhere dense subsets in suitable totally bounded topologies.

It follows from the definition that the family $S$ of small subsets of a group is an invariant ideal. This ideal relates to the other ideals as follows:

$$\mathcal{U} \mathcal{N} \subset S \subset \bigcup_{\mu} \mathcal{N}_\mu.$$ 

**Question:** What can be understood by universally small subset?

**Hint:** We need a counterpart of the countable chain condition for ideals in countable groups.
Packing index

Given a subset $A \subset G$ consider the cardinal

$$\text{pack}(A) = \sup\{|B| : B \subset G \quad \{b + A\}_{b \in B} \text{ is disjoint}\}$$

called the packing index of $A$.

Example: $\text{pack}(2\mathbb{Z}) = 2$. 
Problem (Omiljanowski)

Is it true that the packing index $\text{pack}(A)$ of a Borel subset of $\mathbb{R}$ is either at most countable or else equal to $\mathfrak{c}$.

(This is true if $A$ is $\sigma$-compact.)
Let $I$ is an ideal of subsets of a group. We define a family $A$ of subsets of $G$ to be $I$-disjoint if $A \cap A' \in I$ for any two distinct sets $A, A' \in A$. If $I = \{\emptyset\}$ (resp. $I = [G]^{<\omega}$), then $I$-disjoint is the same as (almost) disjoint in the usual sense.

Introducing an ideal parameter in the definition of a packing index, we obtain the notion of the $I$-packing index

$$I\text{-pack}(A) = \sup\{|B| : B \subset G \{b + A\}_{b \in B} \text{ is } I\text{-disjoint}\}.$$
Definition
An ideal \( I \) on \( G \) is pack-\textit{complete} if each subset \( A \subset G \) with \( I\text{-}\text{pack}(A) \geq \aleph_0 \) belongs to \( I \).

So, the packing completeness can be thought as a counterpart of ccc-property for ideals on countable groups.
Examples of packing complete ideals:

The following ideals are packing complete:

- $\mathcal{N}_\mu$ for every invariant measure $\mu$ on $G$;
- $\mathcal{U}\mathcal{N} = \bigcap_\mu \mathcal{N}_\mu$;
- $S$, the ideal of small subsets of a countable abelian group $G$. 
The packing completion of an ideal

Theorem
For every ideal $\mathcal{I}$ on a countable abelian group $G$ the intersection $\tilde{\mathcal{I}}$ of all packing complete ideals that contain $\mathcal{I}$ is a well-defined packing complete ideal called the packing completion of $\mathcal{I}$. It is equal to the union

$$\tilde{\mathcal{I}} = \bigcup_{\alpha < \omega_1} \mathcal{I}_\alpha$$

where $\mathcal{I}_0 = \mathcal{I}$ and $\mathcal{I}_\alpha$ is the ideal generated by all subsets with infinite $\mathcal{I}_{<\alpha}$-packing index.
The packing completion $\mathcal{US}$ of the empty ideal $\mathcal{I} = \{\emptyset\}$ is the smallest packing complete ideal. So, we get the chain of packing complete ideals:

$$\mathcal{US} \subset \mathcal{UN} \subset S \subset \bigcup_{\mu} \mathcal{N}_{\mu}.$$ 

The last two inclusions cannot be reversed.

Problem

1. Is $\mathcal{US} \neq \mathcal{UN}$?
2. Find a combinatorial characterization of subsets belonging to the ideal $\mathcal{US}$.
3. What is the descriptive complexity of the ideals $\mathcal{US}$ and $\mathcal{UN}$?
Thank you!