A survey on small measures on compact spaces and Boolean algebras

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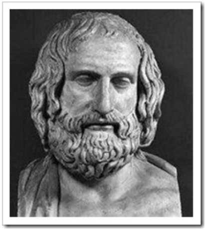
$\Pi\alpha\nu\tau\omega\nu\ \chi\rho\eta\mu\alpha\tau\omega\nu\ \mu\epsilon\tau\rho o\nu\ \alpha\nu\theta\rho\omega\pi o\varsigma$

(Panton chrematon **metron** anthropos)



Protagoras (490 – 420 BC)

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Every *CD* measure has a separable support.

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The type of $\mu \in P(\mathfrak{A})$ is uncountable iff there is $\{a_\xi : \xi < \omega_1\} \subseteq \mathfrak{A}$ such that $\inf_{\xi \neq \eta} \mu(a_\xi \triangle a_\eta) > 0$.

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7. Borodulin-Nadzieja '07: contains Stone spaces of minimally generated Boolean algebras.

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Problem (Roman Pol)

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Theorem (Mikołaj Krupski &GP)

Every compact space either carries a SCD measure or carries a measure of uncountable type.

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- Dow & Pichardo-Mendoza '09: Under CH there is a minimally generated Boolean algebra $\mathfrak{A}$ such that its Stone space K is Efimov.

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Efimov spaces and measures

Definition

A Efimov space is a compact space containing no nontrivial converging sequences and no copy of $\beta\omega$

- K contains no copy of $\beta\omega$ iff K admits no continuous surjection onto $[0,1]^{\mathfrak{c}}$.
- Hence if K contains no converging sequence and every $\mu \in P(K)$ has countable type then K is Efimov.
- Dzamonja & GP '07: Under CH there is such a space K.
- Dow & Pichardo-Mendoza '09: Under CH there is a minimally generated Boolean algebra $\mathfrak{A}$ such that its Stone space K is Efimov. It follows from Borodulin-Nadzieja '07 that every $\mu \in P(K)$ is CD (in fact every nonatomic $\mu \in P(K)$ is SCD).

The topology of $P(K)$

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- (1) generalizes the result on Rosenthal compacta.

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Theorem (Sobota & GP)

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A Banach space has property (C) if for every family $\mathcal{C}$ of closed and convex subsets of X, if $\bigcap \mathcal{C}_0 \neq \emptyset$ for every countable $\mathcal{C}_0 \subseteq \mathcal{C}$ then $\bigcap \mathcal{C} \neq \emptyset$.

Problem (Roman Pol)

Does countable tightness of $P(K)$ imply countable tightness of $P(K \times K)$?