Shelah's Easy Black Box

Gabriel Salazar
Universität Freiburg

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Shelah's Black Box - Brief History

- Combinatorial principle in ZFC.
- Partially predicts maps under cardinal conditions.
- First appeared in 1985 (Udine Conference on Abelian Groups) without an explicit name.
- Gerenal Black Box from A.L.S. Corner and R. Göbel, *Prescribing endomorphism algebras - A unified treatment*.
- Different versions of the Black Box appear, like the Strong Black Box and variations.

- *Easy Black Box* appeared in 2007 (Cubo - A Mathematical Journal).
- More applications in (complicated) algebraic constructions.
- Current state of development: Replace the Black Box by the Easy Black Box and a suitably strong Step Lemma.

Notation and Definitions

Order-preserving sequences

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Order-preserving finite sequences

$$^{\omega\uparrow>}\lambda = \{\, \eta : \ell \to \lambda \mid \eta(m) < \eta(n) \text{ for } m < n < \ell < \omega \,\}.$$

Definition

For $\eta \in {}^{\omega\uparrow}\lambda \cup {}^{\omega\uparrow>}\lambda$, *the* **support** *of* η *is*

$$[\eta] = \{\, \eta \restriction n \mid n \in \mathrm{dom}(\eta) \,\}$$

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Definition

For a set $\mathfrak{X}$, *a* **trap** *is*

$$g_\eta : [\eta] \to \mathfrak{X}.$$

The Easy Black Box

For each cardinal $\lambda \geq \aleph_0$ and set $\mathfrak{X}$ of cardinality $\leq \lambda^{\aleph_0}$ there is a family of traps

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Prediction Principle: for all $g : {}^{\omega\uparrow>}\lambda \to \mathfrak{X}$ and $\nu \in {}^{\omega\uparrow>}\lambda$, we can find $\eta \in {}^{\omega\uparrow}\lambda$ with $\nu \subset \eta$ and $g_\eta \subseteq g$.

Definition

A trap for the Strong Black Box is a quintuple $p = (\eta, V_*, V, \mathfrak{F}, \varphi)$ *such that*

1. $\eta \in {}^{\omega\uparrow}\lambda_k$,
2. $V \in [\Lambda]^{\leq\lambda_{k-1}}$ *and* $V_* \in [\Lambda_*]^{\leq\lambda_{k-1}}$,
3. (V_*, V) *is* Λ*-closed,*
4. $\Lambda^{\eta *} \subseteq V_*$,
5. $\| \overline{\xi} \| < \| \eta \|$ *for all* $\overline{\xi} \in V \cup V_*$,
6. *For* $\overline{\eta} \in \Lambda$, *if* $\| \overline{\eta} \| < \| \eta \|$ *and* $k \notin u_{\overline{\eta}}(V_*)$, *then* $\overline{\eta} \in V$.
7. *For* $\overline{\eta} \in \Lambda$, *if* $([\overline{\eta}] \setminus [\overline{\eta} \upharpoonright k]) \cap V_* \neq \emptyset$, *then* $[\overline{\eta}] \subseteq V_*$.
8. $\mathfrak{F} = \mathfrak{F}_{V_* V} = \{ y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V, b_{\overline{\eta}} \in \overline{B}_{V_*} \}$ *is regressive,*
9. $\varphi : G_{V_* V} \to G_{V_* V}$ *is a homomorphism.*

The Strong Black Box

Let μ be an infinite cardinal, $\lambda = \mu^+$, $\theta \leq \lambda$ such that $\mu^\theta = \mu$ and $k > 1$. If $E \subseteq \lambda^o$ is stationary, then there is a family

$$\{ p_\alpha = (\eta^\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha) \mid \alpha < \lambda \}$$

of traps such that

(1) $\| \eta^\alpha \| \in E$ *for all* $\alpha < \lambda$,

(2) $\| \eta^\alpha \| \leq \| \eta^\beta \|$ *for all* $\alpha < \beta < \lambda$,

(3) *If* $\| \eta^\alpha \| = \| \eta^\beta \|$ *for* $\alpha \neq \beta$, *then* $\| V_{\alpha *} \cap V_{\beta *} \| < \| \eta^\alpha \|$,

(4) For any $\mathcal{V} \subseteq \Lambda$, any regressive family $\mathfrak{F}_{\Lambda_* \mathcal{V}} = \{ y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in \mathcal{V}, b_{\overline{\eta}} \in \overline{B} \}$, any $\varphi \in \mathrm{End}\, G_{\Lambda_* \mathcal{V}}$, $U \in [\Lambda_*]^{\leq \theta}$ and $\delta < \lambda$, the set of $\gamma \in E$ for which there is some $\alpha < \lambda$ with

$$\| \eta^\alpha \| = \gamma,\, \delta < 0\eta^\alpha,\; V_\alpha = \mathcal{V}_{V_{\alpha *}},\, \mathfrak{F}_\alpha = \mathfrak{F}_{\Lambda_* V_\alpha},\, \varphi_\alpha \subseteq \varphi,\, U \subseteq V_{\alpha *}$$

is stationary.

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1. $\beth_0^+(\mu) = \mu^+$.
2. $\beth_{n+1}^+(\mu) = \left(2^{\beth_n^+(\mu)}\right)^+$.

Definition

For a commutative ring R with 1 and a countable multiplicatively closed subset $\mathbb{S} \subset R \setminus \{0\}$ we say that

1. *R is $\mathbb{S}$-**reduced** if $\bigcap_{s \in \mathbb{S}} sR = 0$.*

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2. *R is $\mathbb{S}$-**torsion-free** if $sr = 0$ with $s \in \mathbb{S}$ and $r \in R$ implies $r = 0$.*
3. *R is an $\mathbb{S}$-**ring** if R is $\mathbb{S}$-reduced and $\mathbb{S}$-torsion-free.*

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2. *We say that a R-module is κ-***free** *if subsets of size $< \kappa$ are contained in a free R-submodule.*

Theorem

Let R be a cotorsion-free $\mathbb{S}$-ring and A an R-algebra with $|A| \leq \mu$ and free R-module A_R. If $\lambda = \beth_k^+(\mu)$ for some positive integer k, then we can construct an $\aleph_k$-free A-module G of cardinality λ with R-endomorphism algebra

$$\operatorname{End}_R G = A.$$

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(For example, take $R = \mathbb{Z}$, $\mathbb{S} = \{ p^n \mid n < \omega \}$ for a fixed prime number p and A a ring with free additive structure.)

Motivation

Theorem (A.L.S. Corner)

If a ring R with 1 is

1. *countable,*
2. *reduced ($\bigcap_{r \in R \setminus \{0\}} rR = 0$) and*
3. *torsion-free (as abelian group),*

then

$$R \cong \operatorname{End} G$$

for a countable, reduced, torsion-free abelian group G.

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How to extend this construction to $\aleph_k$-freeness for $k > 1$?

Sketch of Construction

Take $k > 1$, $\mu = 2^{|A|}$ and for $1 \leq m \leq k$ let

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These cardinals satisfy the following *cardinal condition*:

$$\lambda_{m+1}^{\lambda_m} = \lambda_{m+1}$$

for all $1 \leq m < k$.

Sketch of Construction

Consider the following sets:

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and

$$\Lambda_* = \dot{\bigcup_{1 \le m \le k}} \Lambda_m,$$

where

$$\Lambda_m = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_k.$$

Elements of Λ:

$$\overline{\eta} = (\eta_1, \ldots, \eta_k)$$

Elements of Λ_*:

$$\overline{\eta} \upharpoonleft \langle m, n \rangle = (\eta_1, \ldots, \eta_m \upharpoonright n, \ldots, \eta_k)$$

We consider the free A-module

$$B = \bigoplus_{\overline{\nu} \in \Lambda_*} A e_{\overline{\nu}}$$

and its p-completion $\widehat{B}$.

The idea is to choose a family $\mathfrak{F} \subseteq \widehat{B}$ to construct

$$G = \langle B, \mathfrak{F} \rangle_* = \{ b \in \widehat{B} \mid p^n b \in \langle B, \mathfrak{F} \rangle \text{ for some } n < \omega \}$$

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where for all $n < \omega$,

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In this way,

$$B \subseteq G \subseteq \widehat{B}.$$

For $X_* \subseteq \Lambda_*$, you can also consider submodules

$$B_{X_*} = \bigoplus_{\overline{\nu} \in X_*} Ae_{\overline{\nu}}$$

and do the same to obtain an A-module G_{X_*} with

$$B_{X_*} \subseteq G_{X_*} \subseteq \widehat{B}_{X_*}.$$

The family $\mathfrak{F}$ will be of the form

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where

1. $X \subseteq \Lambda$.
2. The elements

$$y_{\overline{\eta}} = \sum_{i=0}^{\infty} p^i \left(\sum_{m=1}^{k} e_{\overline{\eta} \restriction \langle m,i \rangle} \right)$$

are specific, previously constructed elements of $\widehat{B}_{X_*}$.

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3. $b_{\overline{\eta}} \in B_{X_*}$, $\pi_{\overline{\eta}} \in \widehat{R}$.

The Step Lemma

Step Lemmas allow us to choose the elements of $\mathfrak{F}$ in order to eliminate unwanted endomorphisms.

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The BASIC idea is the following:

If an $\mathbb{S}$-ring R satisfies $\pi R \cap R = 0$ for some $\pi \in \widehat{R}$ and you

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3. have an element $z \in B_{X_*}$ with $z\varphi \notin Az$,

then you can choose an $\pi_{\overline{\eta}} \in \{0, \pi\}$ such that φ does not extend to an endomorphism

$$\varphi : \langle B_{X_*}, \pi_{\overline{\eta}} z + y_{\overline{\eta}} \rangle_* \to \langle B_{X_*}, \pi_{\overline{\eta}} z + y_{\overline{\eta}} \rangle_* .$$

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In the proof of this theorem, $\mathfrak{X}$ is a set of tuples

$$(G, H, P, Q, R, \psi)$$

where the entries are either A-submodules or subsets of Λ and Λ_* of size λ_m that belong to families of size $\lambda_{m+1}^{\lambda_m} = \lambda_{m+1}$, and $\psi : G \to H$.

WARNING

The following is an **oversimplified** argument!

The proof goes on induction on $k-1$ starting at 0.

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If we are at stage m of the induction, take an enumeration of ${}^{\omega\uparrow}\lambda_m = \langle\, \eta_\alpha \mid \alpha < \lambda_m \,\rangle$ without repetitions.

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By letting α run and checking trap by trap at

$$g_{\eta_\alpha}(\eta_\alpha \restriction n) = (G_{\alpha n}, H_{\alpha n}, P_{\alpha n}, Q_{\alpha n}, R_{\alpha n}, \psi_{\alpha n}),$$

if these components extend each other and $\psi_{\alpha n}$ coincides with φ in $G_{\alpha n}$, then we choose $\pi_{\overline{\eta}}$ to kill φ. Otherwise just take $\pi_{\overline{\eta}} = 0$.

Question

What else could be constructed with the Easy Black Box?

Thank You!

References

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