Subcontinua of $\mathbb{H}^*$

Quidquid latine dictum sit, altum videtur

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Outline

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Main result

Theorem (Alan Dow, Y. T.)

The Čech-Stone remainder of the half line $\mathbb{H}$ has a family of $2^{\mathfrak{c}}$ many mutually nonhomeomorphic subcontinua.

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The rest of this talk will consist of an explanation of all the terms and of a sketch of the proof.

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Compactification: a compact Hausdorff space that contains (a homeomorphic copy of) X as a dense subspace.

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since when do we give ideas logical names?

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then their closures in $\beta\mathbb{H}$ are disjoint.

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Crucial property

Very nice open sets

Let F and G be closed and disjoint in $\mathbb{H}^*$.

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Let F and G be closed and disjoint in $\mathbb{H}^*$. There are two sequences of open intervals, $\langle (a_n, b_n) : n \in \omega \rangle$ and $\langle (c_n, d_n) : n \in \omega \rangle$, such that

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- $b_n < c_n$ and $d_n < a_{n+1}$ for all n (or the other way round);
- $F \subseteq \operatorname{Ex} U$ and $G \subseteq \operatorname{Ex} V$,

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- $F \subseteq \operatorname{Ex} U$ and $G \subseteq \operatorname{Ex} V$, where $U = \bigcup_n (a_n, b_n)$ and $V = \bigcup_n (c_n, d_n)$.

Notation: $\operatorname{Ex} U = \beta\mathbb{H} \setminus \operatorname{cl}_\beta(\mathbb{H} \setminus U)$

Indication of proof

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Keep alternating

About U and V

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And dually: the sets of the form $\mathbb{H}^* \cap \operatorname{cl}_\beta \bigcup_n [a_n, b_n]$ form a base for the closed sets of $\mathbb{H}^*$.

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Also well known: $\mathbb{H}^*$ contains ω^* and ω^* has $2^{\mathfrak{c}}$ many points, so $\mathbb{H}^*$ has exactly $2^{\mathfrak{c}}$ many points.

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Exercise: a decreasing sequence of compact connected sets has a compact and connected intersection.

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Take $x \in \mathbb{H}^*$.
Let u be the family of subsets, A, of ω that satisfy

$$x \in \mathrm{cl}_\beta \bigcup_{n \in A} [a_n, b_n]$$

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In fact ...

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Repeat previous arguments: if $x \notin K$ find a nice closed set, $F = \bigcup_n [a_n, b_n]$, such that $K \subseteq \operatorname{cl}_\beta F$ and $x \notin \operatorname{cl}_\beta F$.

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As above, $u = \{A : K \subseteq \mathrm{cl}_\beta \bigcup_{n \in A} [a_n, b_n]\}$ is an ultrafilter, $K \subseteq [a_u, b_u]$, and $x \notin [a_u, b_u]$

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Indecomposability

Not too difficult: $[a_u, b_u]$ has empty interior in $\mathbb{H}^*$.

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Corollary: if K and L are two *proper* subcontinua of $\mathbb{H}^*$ then $K \cup L \neq \mathbb{H}^*$.

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Corollary: if K and L are two *proper* subcontinua of $\mathbb{H}^*$ then $K \cup L \neq \mathbb{H}^*$.

In other words: $\mathbb{H}^*$ is an indecomposable continuum. (Bellamy, Woods).

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Standard subcontinua of $\mathbb{H}^*$

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- for $u \in \omega^*$ the set $[a_u, b_u]$ looks like $\beta\pi^{\leftarrow}(u)$
- we write $\mathbb{I}_u$ for this preimage.

Properties

The continuum $\mathbb{I}_u$

- is irreducible between 0_u and 1_u

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The points of $(0,1)^\omega/u$ are cut points of $\mathbb{I}_u$ but . . .

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... if $\langle x_n : n \in \omega \rangle$ is an increasing sequence of such cut points then its 'supremum' is a non-trivial continuum.

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These layers will be important later on.

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Further properties

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Then $[c_v, d_v] \subseteq [a_u, b_u]$ iff the (partial) function

$$\varphi = \{\langle m, n \rangle : [c_m, d_m] \subseteq [a_n, b_n]\}$$

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satisfies $\varphi(v) = u$ (so, implicitly, $\operatorname{dom} \varphi \in v$ and $\operatorname{ran} \varphi \in u$).

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Further properties

Two cases:

- φ is one-to-one on some member of v, then $[c_v, d_v]$ is a subinterval of $[a_u, b_u]$

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Two cases:

- φ is one-to-one on some member of v, then $[c_v, d_v]$ is a subinterval of $[a_u, b_u]$
- φ is one-to-one on no member of v, then $[c_v, d_v]$ is a subset of some layer of $[a_u, b_u]$

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Further properties

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Lemma

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For the proof see the references at the end.

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CH fails

Theorem (Alan Dow, $\neg$CH)

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There exists a family of $2^{\mathfrak{c}}$ mutually non-homeomorphic standard subcontinua.

Proof.

Based on a result of Shelah's on the existence of a family of $2^{\mathfrak{c}}$ mutually non-isomorphic ultrapowers of $(0, 1)$. □

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A byproduct of our construction is a family of $2^{\mathfrak{c}}$ mutually non-homeomorphic decomposable subcontinua.

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Every sequence in Γ gives us a cover of $\mathbb{H}^*$ by standard subcontinua: the family $\{[a_u, b_u] : u \in \omega^*\}$.

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Γ is the set of all sequences $\langle [a_n, b_n] : n \in \omega \rangle$ of closed intervals, with $a_n, b_n \in \omega$ and $a_{n+1} = b_n$ for all n.

Every sequence in Γ gives us a cover of $\mathbb{H}^*$ by standard subcontinua: the family $\{[a_u, b_u] : u \in \omega^*\}$.

If two of these standard subcontinua intersect then it is (only) in the following situation: $b_u = a_v$ and $v = u + 1$.

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If two of these standard subcontinua intersect then it is (only) in the following situation: $b_u = a_v$ and $v = u + 1$. These cases will not really be important in what follows.

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Notation

If $A \in \Gamma$, say $A = \langle [a_n, b_n] : n \in \omega \rangle$, and $u \in \omega^*$ then A_u is the standard subcontinuum from the cover that contains u.

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For most of the A it is actually the case that u is in a layer $L(A, u)$ of A_u; this happens if the map $\{\langle m, n \rangle : m \in [a_n, b_n]\}$ is one-to-one on no member of u.

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By our technical result the $L(A, u)$ form a chain $\mathcal{C}_u$; and this is what we will use.

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Main technical result, from CH

Theorem

For every linearly ordered set T of size at most $\aleph_1$

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For every linearly ordered set T of size at most $\aleph_1$ there are a P-point u and a map $t \mapsto A_t$ from T to Γ such that $t \mapsto L(A_t, u)$ is an embedding of T into $\mathcal{C}_u$.

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Theorem

For every linearly ordered set T of size at most $\aleph_1$ there are a P-point u and a map $t \mapsto A_t$ from T to Γ such that $t \mapsto L(A_t, u)$ is an embedding of T into $\mathcal{C}_u$.
In addition: if T has no $\langle\omega, \omega^\star\rangle$-gaps then we can make sure that $I(T, u) = \{L(A_t, u) : t \in T\}$ is an interval in $\mathcal{C}_u$.

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These we call mean linear orders.

Mean linear orders

Adjoin S as a maximum to S (and ditto for T) and apply our main technical result to the resulting ordered sets to get P-points u and v, and the corresponding embeddings.

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Let us consider the layers $L(A_S, u)$ and $L(A_T, v)$.

Mean linear orders

Because of the interval property the indecomposable continuum $L(A_S, u)$ is the closure of the F_σ-set $\bigcup_{s\in S} L(A_s, u)$ (and likewise for T and v).

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Mean linear orders

Because of the interval property the indecomposable continuum $L(A_S, u)$ is the closure of the F_σ-set $\bigcup_{s\in S} L(A_s, u)$ (and likewise for T and v).

Let $f : L(A_S, u) \to L(A_T, v)$ be a homeomorphism. Because the $L(A_t, u)$ are P-sets we must have $L(A_t, u) \cap f\left[\bigcup_{s\in S} L(A_s, u)\right] \neq \emptyset$ for all t (and vice versa for the $f\left[L(A_s, u)\right]$ and $\bigcup_{t\in T} L(A_t, v)$.

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Use the early technical result to conclude that $f\left[\bigcup_{s\in S} L(A_s, u)\right] = \bigcup_{t\in T} L(A_t, v)$.

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It gets better

We even get, thanks to the interval property again, that the relation

$$\{\langle s,t\rangle : f\left[L(A_s,u)\right] = L(A_t,v)$$

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We even get, thanks to the interval property again, that the relation

$$\bigl\{\langle s,t\rangle : f\bigl[L(A_s,u)\bigr] = L(A_t,v)\bigr\}$$

is an isomorphism between final segments of S and T.

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Many mean linear orders

For a set, X, of limit ordinals in ω_1 insert a decreasing ω-sequence between α and $\alpha+1$ for all $\alpha \in X$, to form L_X, say.

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$T_X = \omega \times L_X$, ordered lexicographically.

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By a happy coincidence $\aleph_1 = \mathfrak{c}$, so we have $2^{\mathfrak{c}}$ mean linear orders without isomorphic final segments.

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In each case take, in the standard continuum A_T, the closed 'interval' $J(A_T, u)$ from one end point to the layer $L(A_T, u)$.

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In each case take, in the standard continuum A_T, the closed ‘interval’ $J(A_T, u)$ from one end point to the layer $L(A_T, u)$.

A homeomorphism between $J(A_T, u)$ and $J(A_S, v)$ must map $L(A_T, u)$ to $L(A_S, v)$, so there.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

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Some set-theory, Stone-Čech, and F-spaces, Topology and Applications, **158** (2011), 1749–1755.

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On subcontinua and continuous images of $\beta\mathbb{R} \setminus \mathbb{R}$, `http://arxiv.org/abs/1401.3132`.

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The Čech-Stone compactification of the Real Line, In *Recent progress in general topology* (1992), 317–352.

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