Partition relations for linear orders in a non-choice context

03E02, 03E60, 05C63

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Winterschool on Abstract Analysis, Section Set Theory & Topology, Hejnice,
Sunday, 26th of January 2014, 10:00-10:35

Notation

$$\alpha \to (\beta, \gamma)^n \textit{ means}$$

$$\forall \chi : [\alpha]^n \longrightarrow 2(\exists B \in [\alpha]^\beta \forall t \in [B]^n \chi(t) = 0$$
$$\vee \exists C \in [\alpha]^\gamma \forall t \in [C]^n \chi(t) = 1).$$

Fact (ZFC)

There is no linear order φ such that $\varphi \to (\omega^*, \omega)^2$.

Proof.

Suppose $\varphi \to (\omega^*, \omega)^2$. Let $<_w$ be a well-order of φ. Let

$$\chi : [\varphi]^2 \longrightarrow 2$$

$$\{x, y\}_< \longmapsto \begin{cases} 0 \text{ iff } x <_w y \\ 1 \text{ else.} \end{cases}$$

⊣

Notation

$$\alpha \to (\beta \vee \gamma, \delta)^n \textit{ means}$$

$$\begin{aligned}\forall \chi : [\alpha]^n \longrightarrow 2(&\exists B \in [\alpha]^\beta \forall t \in [B]^n \chi(t) = 0 \\ &\vee \exists C \in [\alpha]^\gamma \forall t \in [C]^n \chi(t) = 0 \\ &\vee \exists D \in [\alpha]^\delta \forall t \in [D]^n \chi(t) = 1).\end{aligned}$$

Theorem (1971, Erdős, Milner, Rado, ZFC)

There is no order φ *such that* $\varphi \to (\omega^* + \omega, 4)^3$.

Proof.

Well-order φ by $<_w$.

$$\chi : [\varphi]^3 \longrightarrow 2$$
$$\{x, y, z\}_< \longmapsto \begin{cases} 1 \text{ iff } y <_w x, z \\ 0 \text{ else.} \end{cases}$$

⊣

Theorem (1971, Erdős, Milner, Rado, ZFC)

There is no order φ such that $\varphi \to (\omega + \omega^*, 4)^3$.

Proof.

Well-order φ by $<_w$.

$$\chi : [\varphi]^3 \longrightarrow 2$$

$$\{x, y, z\}_< \longmapsto \begin{cases} 1 \text{ iff } x, z <_w y \\ 0 \text{ else.} \end{cases}$$

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Theorem (1971, Erdős, Milner, Rado, ZFC)

There is no order φ such that $\varphi \to (\omega + \omega^* \vee \omega^* + \omega, 5)^3$.

Proof.

Well-order φ by $<_w$.

$$\chi : [\varphi]^3 \longrightarrow 2$$
$$\{x, y, z\}_< \longmapsto \begin{cases} 0 \text{ iff } x <_w y <_w z \vee z <_w y <_w x \\ 1 \text{ else.} \end{cases}$$

⊣

Question (1971, Erdős, Milner, Rado, ZFC)

Is there an order φ *such that* $\varphi \to (\omega + \omega^* \vee \omega^* + \omega, 4)^3$*?*

Theorem (1981, Blass, ZF)

For every continuous colouring χ with $\mathrm{dom}(\chi) = [{}^{\omega}2]^n$ there is a perfect $P \subset {}^{\omega}2$ on which the value of χ at an n-tuple is decided by its splitting type.

Definition

The *splitting type* of an n-tuple $\{x_0, \ldots, x_{n-1}\}_{<_{\mathrm{lex}}}$ is given by the permutation π of $n-1$ such that $\langle \triangle(x_{\pi(i)}, x_{\pi(i)+1}) | i < n-1 \rangle$ is ascending. $\triangle(x, y) := \min\{\alpha | x(\alpha) \neq y(\alpha)\}$.

Remark

For an n-tuple there are $(n-1)!$ splitting types.

Theorem (1981, Blass, ZF)

For every Baire colouring χ with $\mathrm{dom}(\chi) = [{}^{\omega}2]^n$ there is a perfect $P \subset {}^{\omega}2$ on which the value of χ at an n-tuple is decided by its splitting type.

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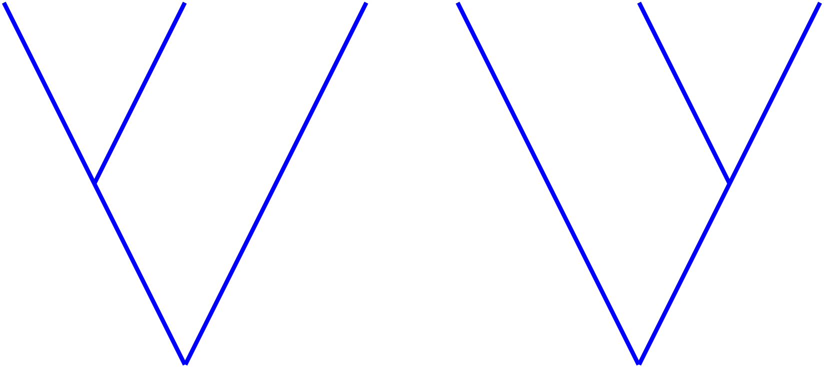
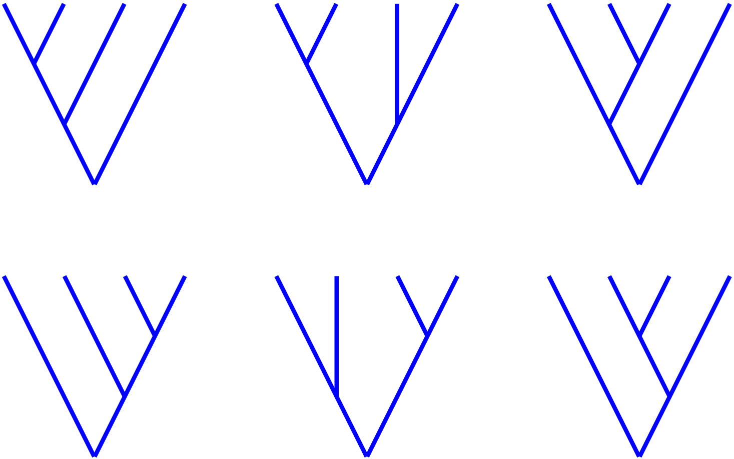
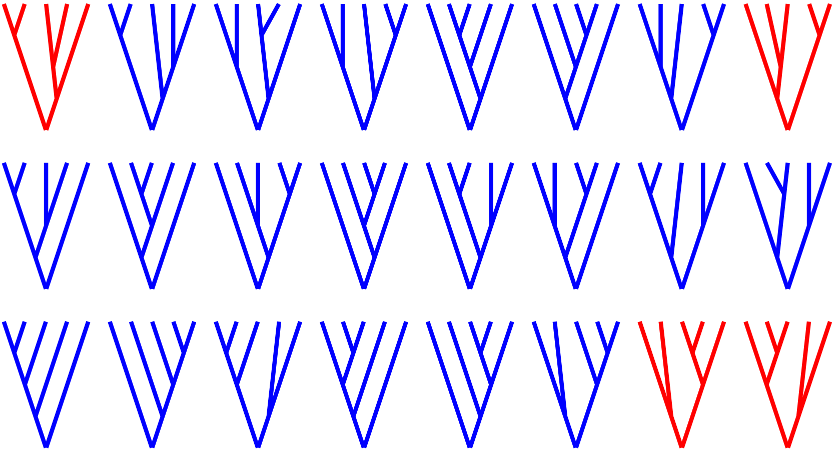
Remark

For an n-tuple there are $(n-1)!$ splitting types.







Axiom (1962, Mycielski, Steinhaus)

(AD): *Every two-player-game with natural-number-moves and perfect information of length ω is determined.*

Axiom (1962, Mycielski, Steinhaus)

($AD_{\mathbb{R}}$): *Every two-player-game with real-number-moves and perfect information of length ω is determined.*

Observation (BP)

$\langle {}^{\omega}2, <_{lex} \rangle \to (\langle {}^{\omega}2, <_{lex} \rangle)^2_2.$

Observation (ZF)

There is no ordinal number α *such that* $\langle {}^{\alpha}2, <_{lex} \rangle \to (\omega^*, \omega)^3.$

Proposition (ZF + BP)

$\langle {}^{\omega}2, <_{lex} \rangle \to (\langle {}^{\omega}2, <_{lex} \rangle, 1 + \omega^* \vee \omega + 1)^3_2.$

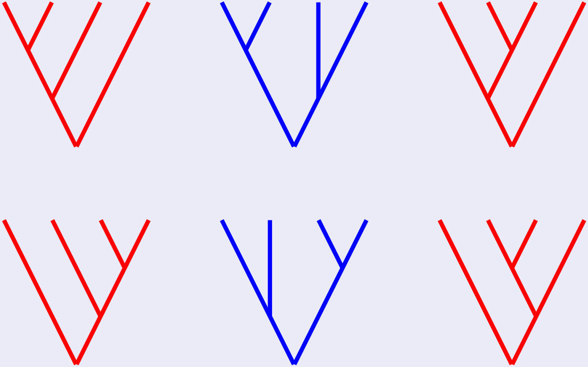
Theorem (ZF)

There is no countable ordinal α *such that* $\langle {}^{\alpha}2, <_{lex} \rangle \to ({}^{\omega+1}2, \aleph_0)^3.$

Theorem (2013, W., ZF)

There is no ordinal number α *such that* $\langle {}^{\alpha}2, <_{lex} \rangle \to (\omega^* + \omega, 5)^4$.

Proof.



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Theorem (2013, W., ZF)

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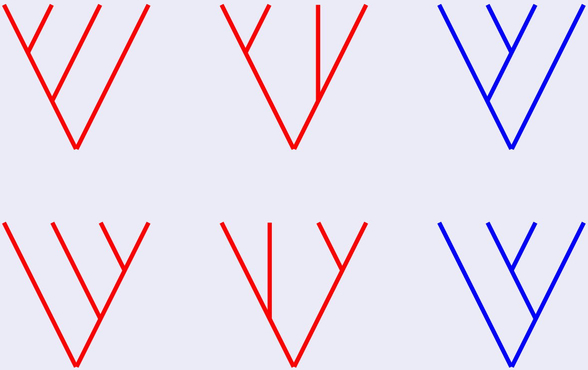
Proof.



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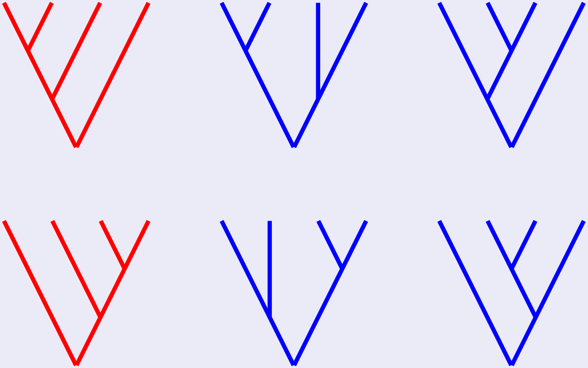
Theorem (2013, W., ZF)

There is no ordinal number α such that $\langle {}^{\alpha}2, <_{lex}\rangle \to (\omega + \omega^* \vee \omega^* + \omega, 7)^4$.

Proof.



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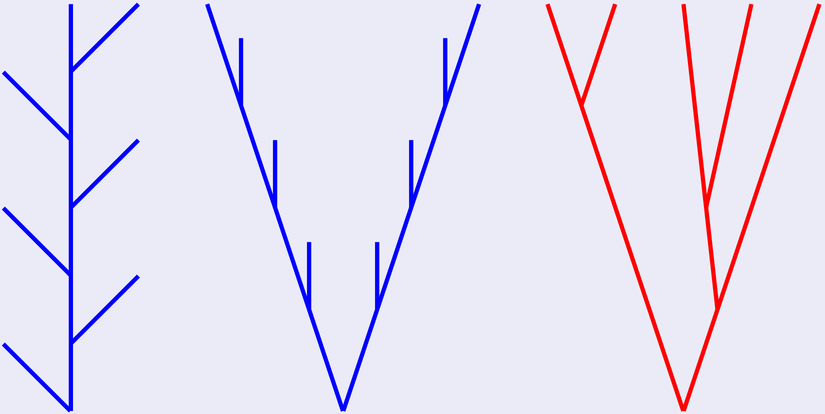
Theorem (2013, W., BP)

$\langle {}^{\omega}2, <_{lex} \rangle \rightarrow (\omega + 1 + \omega^* \vee 1 + \omega^* + \omega + 1, 5)^4.$

Proof.



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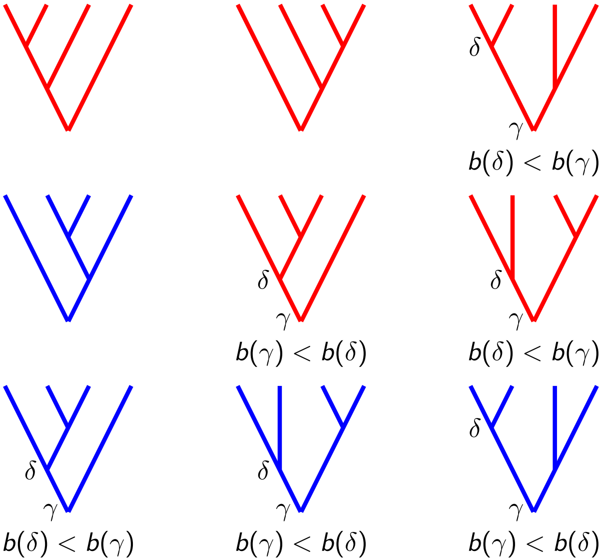
Theorem (2013, W., ZF)

There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{lex} \rangle \rightarrow (\omega + \omega^* \vee \omega^* + \omega, 6)^4.$$


δ
γ
b(δ) < b(γ)
δ
γ
b(γ) < b(δ)
δ
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b(δ) < b(γ)
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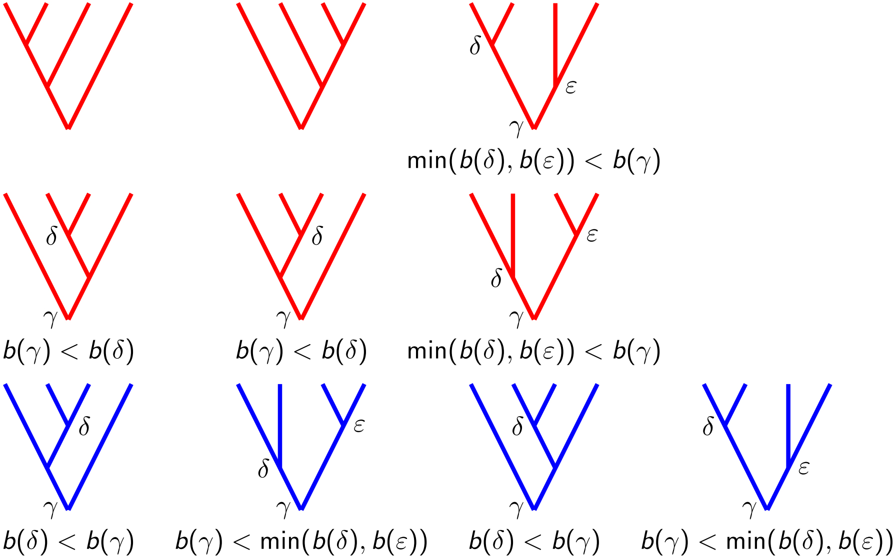
b(δ) < b(γ)
δ
γ
b(γ) < b(δ)
δ
γ
b(γ) < b(δ)

Theorem (2013, W., ZF)

There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{lex} \rangle \rightarrow (\omega + 2 + \omega^* \vee \omega^* + \omega, 5)^4.$$


δ
ε
γ
min(b(δ), b(ε)) < b(γ)
δ
γ
b(γ) < b(δ)
δ
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b(γ) < b(δ)
ε
δ
γ
min(b(δ), b(ε)) < b(γ)
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b(δ) < b(γ)
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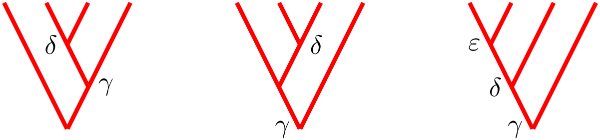
γ
b(γ) < min(b(δ), b(ε))
δ
γ
b(δ) < b(γ)
δ
ε
γ
b(γ) < min(b(δ), b(ε))

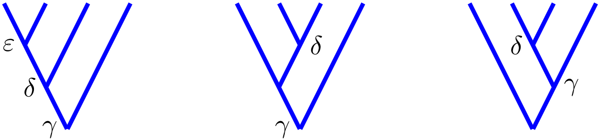
Theorem (2013, W., ZF)

There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{lex} \rangle \to (\omega + \omega^* \vee 2 + \omega^* + \omega, 5)^4.$$



$b(\delta) < b(\gamma)$ $\quad$ $b(\delta) < b(\gamma)$ $\quad$ $b(\delta) < \max(b(\gamma), b(\varepsilon))$



$\max(b(\gamma), b(\varepsilon)) < b(\delta)$ $\quad$ $b(\gamma) < b(\delta)$ $\quad$ $b(\gamma) < b(\delta)$

Theorem (1964, Mycielski, $\mathsf{ZF} + \mathsf{AD}$)

BP.

Theorem (Martin, $\mathsf{ZF} + \mathsf{AD}$)

$\omega_1 \to (\omega_1)^{\omega_1}_{2^{\aleph_0}}$.

Theorem (1976, Prikry, $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$)

$\omega \to (\omega)^{\omega}_2$

Conjecture (2013, W., $ZF + AD_{\mathbb{R}}$)

$\langle {}^{\omega_1}2, <_{lex} \rangle \to (\omega + \omega^* \vee \omega^* + \omega, 6)^4$.

Conjecture (2013, W., $ZF + AD_{\mathbb{R}}$)

$\langle {}^{\omega_1}2, <_{lex} \rangle \to (\omega + 2 + \omega^* \vee \omega^* + \omega, 5)^4$.

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Thank you very much
for your attention!