Asymptotic Density in Generic Extensions

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January 26, 2014
1. Asymptotic density

2. $\mathcal{P}(\mathbb{N})/\mathcal{Z}$ as forcing notion

3. Cardinal invariants and selective ultrafilter in extension
For $A \subseteq \mathbb{N}$ we define upper and lower asymptotic density of $A$ by this formulas

\[
d^*(A) = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n}
\]

\[
d_*(A) = \liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n}.
\]

Let $\mathcal{D} = \{ A : d^*(A) = d_*(A) \}$. We define asymptotic density for $A \in \mathcal{D}$ by

\[
d(A) = \lim_{n \to \infty} \frac{|A \cap [1, n]|}{n}.
\]
$\mathcal{D} \subseteq \mathcal{P}(\mathbb{N})$ but there are sets which does not have asymptotic density.

**Example**

For $i \in \mathbb{N}$ let $A_i = \{ n : \text{first digit of } n \text{ is } i \}$ then $A_i \notin \mathcal{D}$ because $d^*(A_i) = \frac{1}{9i}$ and $d^*(A_i) = \frac{10}{9(i+1)}$.

In fact $\mathcal{D}$ is not even a subalgebra of $\mathcal{P}(\mathbb{N})$ because it is not closed under unions.

**Fact**

If $A, B \in \mathcal{D}$ and $A \cap B = \emptyset$ then $A \cup B \in \mathcal{D}$ and $d(A \cup B) = d(A) + d(B)$.

Even more there exists a measure $\overline{d}$ on $\mathcal{P}(\mathbb{N})$ with $\overline{d} \restriction_{\mathcal{D}} = d$. 
Proposition

Let \( s = \{ a_i : i \in \mathbb{N} \} \) be a sequence of natural numbers such that for each \( i \neq j \) greatest common divisor \( gcd(a_i, a_j) = 1 \). Then the set

\[
A_s = \{ n : \forall i \in \mathbb{N} \ a_i \nmid n \}
\]

has asymptotic density and

\[
d(A_s) = \prod_{i \in \mathbb{N}} \left(1 - \frac{1}{a_i}\right).
\]
Example 1

For $k > 1$ let $P_k = \{n : \forall p \ p^k \not| n\}$ then $P_k = A_{s_k}$ where $s_k = \{p_i^k\}$ where $p_i$ is the $i$-th prime number. Then by Euler formula

$$d(P_k) = \prod_{i \in \mathbb{N}} (1 - \frac{1}{p_i^k}) = \frac{1}{\sum_{n \in \mathbb{N}} \frac{1}{n^k}} = \frac{1}{\zeta(k)},$$

where $\zeta$ is Riemann zeta function.

Example 2

$P_2$ is called set of squarefree numbers and $d(P_2) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$.

Let $\alpha \in \{0, 1\}$ and $P_2^\alpha = \{k : k = p_{i_1}...p_{i_m} \text{ and } m \equiv \alpha \mod 2\}$ then

$$d(P_2^\alpha) = \frac{3}{\pi^2}.$$
**Definition**

\[ \mathcal{Z} = \{ A : A \subseteq \mathbb{N}, d(A) = 0 \} \]

\[ \mathcal{Fin} = \{ A : A \subseteq \mathbb{N}, |A| < \omega \} \]

**Theorem (???)**

\[ \mathcal{P}(\mathbb{N})/\mathcal{Z} \cong \mathcal{P}(\mathbb{N})/\mathcal{Fin} \ast \mathcal{B}(c) \]
Proof, sketch of first part

Let $l_n = [2^n, 2^{n+1})$ and $X \subseteq \mathbb{N}$. Define $f : \mathcal{P}(\mathbb{N})/\mathcal{F}in \to \mathcal{P}(\mathbb{N})/\mathcal{Z}$ with following formula

$$f([X]_{\mathcal{F}in}) = \left[ \bigcup_{n \in X} l_n \right]_{\mathcal{Z}}.$$ 

Then since

$$X \in \mathcal{Z} \iff \lim_{n \to \infty} \frac{X \cap [2^n, 2^{n+1})}{2^n} = 0$$

$f$ is regular embedding.
First question about generic extension is size of $\mathfrak{c}$.

**Proposition**

Forcing with $\mathcal{P}(\mathbb{N})/\mathcal{F}in$ collapses $\mathfrak{c}$ to $\mathfrak{h}$, where $\mathfrak{h}$ is distributivity number of $\mathcal{P}(\mathbb{N})/\mathcal{F}in$.

$\mathbb{B}(\kappa)$ is c.c.c. so does not collaps cardinals.

All together $\mathfrak{c}$ in extension is $\mathfrak{h}$ in groundmodel.
An ultrafilter $\mathcal{U}$ is selective if for every partition $\{J_n\}_{n \in \omega}$ of $\mathbb{N}$ there exists $k \in \omega$ such that $J_k \in \mathcal{U}$ or there exist $U \in \mathcal{U}$ such that $|U \cap J_n| = 1$ for every $n \in \omega$.

**Question**

Is there a selective ultrafilter in extension by $\mathcal{P}(\mathbb{N})/\mathcal{Z}$??

**Answer**

If $\mathfrak{h} = \omega_1$ in $V$ then we can construct a selective ultrafilter because $CH$ is true in $V[G]$. Otherwise we do not know.
Fact
Because of $\mathbb{B}(c) \, \text{cov}(\mathcal{M}) = \text{non}(\mathcal{N}) = \omega_1$ in extension.

Theorem (Canjar)
Every filter generated by $< c$ elements can be extended to a selective ultrafilter iff $\text{cov}(\mathcal{M}) = c$.

Generic filter over $\mathcal{P}(\mathbb{N})/\mathcal{F}in$ is selective in extension.

Theorem (Kunen)
Selective ultrafilter in $V$ cannot be extended to $\mathcal{P}$-(selective) ultrafilter in $V[G]$ where $G$ is generic over $\mathbb{B}(\kappa)$.
Forcing with $\mathbb{B}(\kappa)$ where $\kappa > c$ creates universe without selective ultrafilters.

What about $\mathbb{B}(\kappa)$ where $\kappa \leq c$??
Asymptotic density

Proposition

In \( V[G] \) \( \mathfrak{h} = \omega_1 \) and because of that

\[
V^{\mathcal{P}(\mathbb{N})/\mathcal{Z}} \models \mathcal{P}(\mathbb{N})/\mathcal{Z} \models \text{CH}.
\]

In other words there are selective ultrafilters after two step iteration of \( \mathcal{P}(\mathbb{N})/\mathcal{Z} \).
Thank you.