Finite Compactifications of $\omega^* \setminus \{x\}$

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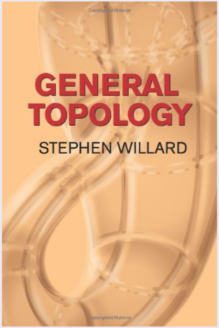
Contents

How this project started

Willard's book on Topology and a curious exercise about the Cantor set

Willard's book



Exercise 30.C

- Show that every open subset of the Cantor set C is homeomorphic either to C or to $C \setminus \{0\}$
- Proof uses Brouwer's characterisation: C is the unique zero-dim. compact metric space without isolated points

A first application

An alternative characterisation of the Cantor set

Definition (Diversity of a space)

The number of nonempty open subsets, up to homeomorphism, of a topological space X is called the *diversity of* X.

- Studied by Rajagopalan/Franklin '90 and Norden/Purisch/Rajagopalan '96.
- The Cantor set is compact of diversity 2.
- The Double Arrow is another example of a compact space of diversity $2 +$ many more.

Theorem (Gruenhage/Schoenfeld '75)

The Cantor set is topologically the unique compact metric space of diversity 2.

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A second application

Finite compactifications of $C \setminus \{0\}$ are all homeomorphic

Theorem

The space $C \setminus \{0\}$ has arbitrarily large finite compactifications.

Theorem

All finite compactifications of $C \setminus \{0\}$ are homeomorphic to C.

Proof strategy:

- Either directly apply Brouwer's characterisation
- or choose a divide-and-conquer tactic

A second application

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A framework for self-similar finite compactifications

The essence that made divide-and-conquer work

Lemma

Let X be a zero-dimensional compact Hausdorff space such that $X \oplus X$ is homeomorphic to X and for some point x of X

$(\star)$ *the one-point compactification of every clopen non-compact subset $A \subset X \setminus \{x\}$ is homeomorphic to X.*

Under these conditions, all finite compactifications of $X \setminus \{x\}$ are homeomorphic to X.

- Applies to all infinite compact Hausdorff spaces of diversity 2...
- ...and to ω^*.

The Stone-Čech remainder ω^* of the integers

A topological characterisation requiring the Continuum Hypothesis

- The Stone-Čech remainder ω^* is the space $\beta\omega \setminus \omega$.
- It is compact and zero-dimensional; disjoint open F_σ-sets have disjoint closures; non-empty G_δ-sets have infinite interior.
- A space with these properties is called *Parovičenko space*.

Theorem (Parovičenko '63; van Douwen/van Mill '78)

[CH] is equivalent to the assertion that every Parovičenko space of weight $\mathfrak{c}$ is homeomorphic to ω^*.

Finite compactifications of $\omega^* \setminus \{x\}$

Many non-equivalent finite compactifications, but they are all homeomorphic

Theorem

[CH]. *Any space $\omega^* \setminus \{x\}$ has arbitrarily large N-point compactifications.*

Theorem

[CH]. *All finite compactifications of $\omega^* \setminus \{x\}$ are homeomorphic to ω^* such that at most one point in the remainder is a non-P-point.*

- Parovičenko space: compact and zero-dimensional; Disjoint open F_σ-sets have disjoint closures; Non-empty G_δ-sets have infinite interior.
- A point $p \in \omega^*$ is a P-point if $p \notin \partial U$ for all open F_σ-sets U of ω^*.

The κ-Parovičenko spaces of weight $\kappa^{<\kappa}$

A common generalisation of C and ω^* to higher cardinals

- κ-Parovičenko space: compact and zero-dimensional; Disjoint open $F_{<\kappa}$-sets have disjoint closures; Non-empty $G_{<\kappa}$-sets have infinite interior.

Brouwer 1910: C

There is a unique zero-dim. cpt. space of weight ω without isolated points.

Parovičenko '63: ω^*

Under [CH] there is a unique Parovičenko space of weight $\mathfrak{c} = \omega_1$.

Negrepontis '69: S_κ

Under the assumption $\kappa = \kappa^{<\kappa}$ there is a unique κ-Parovičenko space of weight κ.

- It follows that $S_\omega = C$ and under [CH] that $S_{w_1} = \omega^*$.

Finite compactifications of $S_\kappa \setminus \{x\}$

Again: many non-equivalent finite compactifications, but they are all homeomorphic

Theorem

Let $\kappa = \kappa^{<\kappa}$. Any space $S_\kappa \setminus \{x\}$ has arbitrarily large N-point compactifications.

Theorem

Let $\kappa = \kappa^{<\kappa}$. All finite compactifications of $S_\kappa \setminus \{x\}$ are homeomorphic to S_κ such that at most one point in the remainder is a non-P_κ-point.

- A point $p \in S_\kappa$ is a P_κ-point if its neighbourhood filter is $< \kappa$-complete.

Further questions

Question

Is the Cantor set X the unique compact metrizable space such that $X \setminus \{x\}$ has self-similar compactifications for all x?

- One would need to aim for zero-dimensionality.

Question

Find a characterisation for self-similar compactifications. Is property $(\star)$ necessary?

Question

Is it consistent that there is a finite compactification of $\omega^* \setminus \{x\}$ that is not homeomorphic to ω^*?

- It is a Parovičenko space of weight $\mathfrak{c}$ containing a P-point.