Finite Compactifications of $\omega^* \setminus \{x\}$

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How this project started
Willard’s book on Topology and a curious exercise about the Cantor set

Willard’s book

Exercise 30.C
- Show that every open subset of the Cantor set $C$ is homeomorphic either to $C$ or to $C \setminus \{0\}$
- Proof uses Brouwer’s characterisation: $C$ is the unique zero-dim. compact metric space without isolated points
A first application
An alternative characterisation of the Cantor set

Definition (Diversity of a space)
The number of nonempty open subsets, up to homeomorphism, of a topological space $X$ is called the diversity of $X$.

- Studied by Rajagopalan/Franklin ’90 and Norden/Purisch/Rajagopalan ’96.
- The Cantor set is compact of diversity 2.
- The Double Arrow is another example of a compact space of diversity 2 + many more.

Theorem (Gruenhage/Schoenfeld ’75)
The Cantor set is topologically the unique compact metric space of diversity 2.
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Theorem (Gruenhage/Schoenfeld ’75)
*The Cantor set is topologically the unique compact metric space of diversity 2.*
A second application

Finite compactifications of $C \setminus \{0\}$ are all homeomorphic

Theorem

The space $C \setminus \{0\}$ has arbitrarily large finite compactifications.

Theorem

All finite compactifications of $C \setminus \{0\}$ are homeomorphic to $C$.

Proof strategy:

- Either directly apply Brouwer’s characterisation
- or choose a divide-and-conquer tactic
A second application

Finite compactifications of $C \setminus \{0\}$ are all homeomorphic

Theorem

*The space $C \setminus \{0\}$ has arbitrarily large finite compactifications.*

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A framework for self-similar finite compactifications

The essence that made divide-and-conquer work

**Lemma**

Let $X$ be a zero-dimensional compact Hausdorff space such that $X \oplus X$ is homeomorphic to $X$ and for some point $x$ of $X$

$(\star)$ the one-point compactification of every clopen non-compact subset $A \subset X \setminus \{x\}$ is homeomorphic to $X$.

Under these conditions, all finite compactifications of $X \setminus \{x\}$ are homeomorphic to $X$.

- Applies to all infinite compact Hausdorff spaces of diversity 2...
- ...and to $\omega^*$. 
The Stone-Čech remainder $\omega^*$ of the integers
A topological characterisation requiring the Continuum Hypothesis

- The Stone-Čech remainder $\omega^*$ is the space $\beta\omega \setminus \omega$.
- It is compact and zero-dimensional; disjoint open $F_\sigma$-sets have disjoint closures; non-empty $G_\delta$-sets have infinite interior.
- A space with these properties is called Parovičenko space.

**Theorem (Parovičenko ’63; van Douwen/van Mill ’78)**

[CH] is equivalent to the assertion that every Parovičenko space of weight $\mathfrak{c}$ is homeomorphic to $\omega^*$.
Finite compactifications of $\omega^* \setminus \{x\}$

Many non-equivalent finite compactifications, but they are all homeomorphic.

**Theorem**

[CH]. Any space $\omega^* \setminus \{x\}$ has arbitrarily large $N$-point compactifications.

**Theorem**

[CH]. All finite compactifications of $\omega^* \setminus \{x\}$ are homeomorphic to $\omega^*$ such that at most one point in the remainder is a non-$P$-point.

- Parovičenko space: compact and zero-dimensional; Disjoint open $F_\sigma$-sets have disjoint closures; Non-empty $G_\delta$-sets have infinite interior.
- A point $p \in \omega^*$ is a $P$-point if $p \notin \partial U$ for all open $F_\sigma$-sets $U$ of $\omega^*$.
The $\kappa$-Parovičenko spaces of weight $\kappa^{<\kappa}$

A common generalisation of $C$ and $\omega^*$ to higher cardinals

- $\kappa$-Parovičenko space: compact and zero-dimensional; Disjoint open $F_{<\kappa}$-sets have disjoint closures; Non-empty $G_{<\kappa}$-sets have infinite interior.

Brouwer 1910: $C$
There is a unique zero-dim. cpt. space of weight $\omega$ without isolated points.

Parovičenko '63: $\omega^*$
Under [CH] there is a unique Parovičenko space of weight $\mathfrak{c} = \omega_1$.

Negrepontis '69: $S_\kappa$
Under the assumption $\kappa = \kappa^{<\kappa}$ there is a unique $\kappa$-Parovičenko space of weight $\kappa$.

- It follows that $S_\omega = C$ and under [CH] that $S_{\omega_1} = \omega^*$.
Finite compactifications of $S_\kappa \setminus \{x\}$

Again: many non-equivalent finite compactifications, but they are all homeomorphic

**Theorem**

Let $\kappa = \kappa^{<\kappa}$. Any space $S_\kappa \setminus \{x\}$ has arbitrarily large $N$-point compactifications.

**Theorem**

Let $\kappa = \kappa^{<\kappa}$. All finite compactifications of $S_\kappa \setminus \{x\}$ are homeomorphic to $S_\kappa$ such that at most one point in the remainder is a non-$P_\kappa$-point.

- A point $p \in S_\kappa$ is a $P_\kappa$-point if its neighbourhood filter is $<\kappa$-complete.
Further questions

Question

*Is the Cantor set $X$ the unique compact metrizable space such that $X \setminus \{x\}$ has self-similar compactifications for all $x$?*

- One would need to aim for zero-dimensionality.

Question

*Find a characterisation for self-similar compactifications. Is property ($\star$) necessary?*

Question

*Is it consistent that there is a finite compactification of $\omega^* \setminus \{x\}$ that is not homeomorphic to $\omega^*$?*

- It is a Parovičenko space of weight $\mathfrak{c}$ containing a $P$-point.