Examples concerning iterated forcing

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Motivation: We will sketch the proof of the relative consistency (assuming the existence of a strongly inaccessible cardinal) of MA + \( \neg \text{CH} \) + There is no Kurepa tree
MA = For every c.c.c. partial order $P$ and a family $\mathcal{F}$ of cardinality $< 2^{\omega}$ of dense subsets of $P$ there is a filter $G \subseteq P$ such that $D \cap G \neq \emptyset$ for all $D \in \mathcal{F}$
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7. Kurepa tree = $\omega_1$-tree with more than $\omega_1$ uncountable branches
Outline

1. On iterations of forcings
2. On Suslin-free forcings
3. The consistency of MA + ¬CH + There is no Kurepa tree
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Iterated forcing

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Outline

1. On iterations of forcings
2. On Suslin-free forcings
3. The consistency of MA + ¬CH + There is no Kurepa tree
Iterations of forcings of length $\alpha$ are sets of sequences of length $\alpha$. 

If $P_\alpha$ is an iteration of length $\alpha$ and $\dot{Q}_\alpha$ is a $P_\alpha$-name for an atomless partial order, then we define the iteration $P_\alpha \ast \dot{Q}_\alpha$ of length $\alpha + 1$. 

$p \ast \dot{q} \in P_\alpha \ast \dot{Q}_\alpha$ iff $p \in P_\alpha$ and $p \parallel \dot{q} \in \dot{Q}_\alpha$.

$p \ast \dot{q} \leq p' \ast \dot{q}'$ iff $p \leq P_\alpha p'$ and $p \parallel \dot{q} \leq \dot{Q}_\alpha \dot{q}'$.

If $P_\alpha'$ are iterations of lengths $\alpha'$ respectively and $P_\alpha' | \alpha'' = P_\alpha''$ for all $\alpha'' < \alpha' < \alpha$, then we define the iteration $P_\alpha$ of length $\alpha$ with supports $\kappa$:

$p \in P_\alpha$ iff $\forall \alpha' < \alpha$, $p | \alpha' \in P_\alpha'$ and $\text{supp}(p) = \{ \alpha' < \alpha : p(\alpha') \neq 1 \dot{Q}_\alpha \}$ has cardinality $< \kappa$.
1. Iterations of forcings of length $\alpha$ are sets of sequences of length $\alpha$.

2. Iterations of forcings $P_0$ of length 1 are just forcings.
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3. If $P_\alpha$ is an iteration of length $\alpha$ and $\dot{Q}_\alpha$ is a $P_\alpha$-name for an atomless partial order, then we define the iteration $P_\alpha * \dot{Q}_\alpha$ of length $\alpha + 1$. 
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4. $p \bowtie q \in P_\alpha \ast \dot{Q}_\alpha$ iff $p \in P_\alpha$ and $p \models q \in \dot{Q}_\alpha$. 
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4. $p \leq \dot{q} \in P_\alpha \ast \dot{Q}_\alpha$ iff $p \in P_\alpha$ and $p \Vdash \dot{q} \in \dot{Q}_\alpha$.

5. $p \leq \dot{q} \preceq p' \preceq \dot{q}'$ iff $p \leq_{P_\alpha} p'$ and $p \Vdash \dot{q} \preceq \dot{Q}_\alpha \dot{q}'$.
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4. $p \blacktriangledown \dot{q} \in P_\alpha * \dot{Q}_\alpha$ iff $p \in P_\alpha$ and $p \forces \dot{q} \in \dot{Q}_\alpha$.

5. $p \blacktriangleleft \dot{q} \leq p' \blacktriangleleft \dot{q}'$ iff $p \leq P_\alpha p'$ and $p \forces \dot{q} \leq \dot{Q}_\alpha \dot{q}'$.

6. If $P_{\alpha'}$'s are iterations of lengths $\alpha'$ respectively and $P_{\alpha'}|_{\alpha''} = P_{\alpha''}$ for all $\alpha'' < \alpha' < \alpha$ then we define the iteration $P_\alpha$ of length $\alpha$ with supports $< \kappa$:

   $$p \in P_\alpha \text{ iff } \forall \alpha' < \alpha \ p|_{\alpha'} \in P_{\alpha'}$$

   $$\text{supp}(p) = \{ \alpha' < \alpha : p(\alpha') \neq 1_{\dot{Q}_\alpha} \} \text{ has cardinality } < \kappa$$
Observations A:

1. We identify $P_\beta$ for $\beta < \alpha$ with a suborder of $P_\alpha$. Also $P_\beta$-names correspond to some $P_\alpha$-names.
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3. If $D$ is dense in $P_\beta$ then $P_\alpha$ forces that

$$\dot{G}\upharpoonright \beta = \{p\upharpoonright \beta : p \in \dot{G}\} \cap \dot{D} \neq \emptyset$$
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2. For each $\beta < \alpha$ the iteration $P_\alpha$ is equivalent to $P_\beta^* P_{[\alpha, \beta)}$ where $P_{[\alpha, \beta)}$ is an appropriate iteration.

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4. If $\hat{D}$ is a $P_\beta$-name for a dense subset of $\hat{Q}_\beta$, then $P_\alpha$ forces that

   $$\hat{G}(\beta) = \{ p(\beta) : p \in \hat{G} \} \cap \hat{D} \neq \emptyset$$
Observations B:

1. If $P_\beta \parallel \dot{Q}_\beta$ is c.c.c for each $\beta < \alpha$, and $P_\alpha$ is an iteration with finite support, then $P_\alpha$ is c.c.c.
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1. If $P_\beta \parallel \check{Q}_\beta$ is c.c.c for each $\beta < \alpha$, and $P_\alpha$ is an iteration with finite support, then $P_\alpha$ is c.c.c.

2. But there could be $P_1, Q_1$ both c.c.c. such that $P_1^* \check{Q}_1$ is not c.c.c. (because $P_1 \parallel \check{Q}_1$ is c.c.c.)
Observations B:

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3. If $P$ is reversed Suslin tree then $P$ is c.c.c. but $P \ast \check{P}$ is not c.c.c. because $P \times P \subseteq P \ast \check{P}$ is not c.c.c.
Observations C:

1. In general if $\dot{x}$ is a $P_\alpha$-name for $\alpha$ a limit ordinal of (large) cofinality there may not be $\beta < \alpha$ and a $P_\beta$-name $\dot{y}$ such that $P_\alpha \models \dot{x} = \dot{y}$
Observations C:

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2. Let $\kappa$ be a cardinal. Let $P_\alpha$ be an iteration with finite supports of c.c.c. forcings where $\kappa < \text{cf}(\alpha)$ is uncountable. If $P_\alpha \models \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a $P_\beta$-name $\dot{y}$ such that $P_\alpha \models \dot{x} = \dot{y}$
Theorem

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1. If $A \subseteq P_\alpha$ is an antichain, then $\bigcup \{\text{supp}(p) : p \in A\}$ is bounded in $\alpha$. 
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Theorem

Let $\kappa$ be a cardinal. Let $P_\alpha$ be an iteration with finite supports of c.c.c. forcings where $\kappa < \text{cf}(\alpha)$ is uncountable. If $P_\alpha \Vdash \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a $P_\beta$-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

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3. Define $\dot{y} = \bigcup_{\xi \in \kappa} \{ \check{\xi} \} \times A_\xi$
Theorem (GCH) There is a finite support iteration $P_\kappa$ of length $\omega_2$ of c.c.c. forcings such that $P_{\omega_2} \models MA + 2^\omega = \omega_2$
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Do the right book-keeping so that whenever $P_{\omega_2} \models |\dot{P}| \leq \omega_1$ and $P_{\omega_2} \models \dot{P}$ is c.c.c., and $\{\dot{D}_\xi : \xi < \omega_1\}$ are $P_{\omega_2}$-names for dense sets of $\dot{P}$ then there is $\beta < \omega_2$ such that $P_\beta \models \dot{P} = \dot{Q}_\beta$ and there are $P_\beta$-names $\{\dot{E}_\xi : \xi < \omega_1\}$ such that $P_\beta \models \dot{E}_\xi = \dot{D}_\xi$ for $\xi < \omega_1$
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$$\dot{G}(\beta) = \{p(\beta) : p \in \dot{G}\}$$

is a filter in $\dot{P} = \dot{Q}_\beta$ meeting all $\dot{E}_\xi = \dot{D}_\xi$. 

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Theorem

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Do the right book-keeping so that whenever $P_{\omega_2} \forces |\dot{P}| \leq \omega_1$ and $P_{\omega_2} \forces \dot{P}$ is c.c.c., and $\{\dot{D}_\xi : \xi < \omega_1\}$ are $P_{\omega_2}$-names for dense sets of $\dot{P}$ then there is $\beta < \omega_2$ such that $P_\beta \forces \dot{P} = \dot{Q}_\beta$ and there are $P_\beta$-names $\{\dot{E}_\xi : \xi < \omega_1\}$ such that $P_\beta \forces \dot{E}_\xi = \dot{D}_\xi$ for $\xi < \omega_1$

Then $P_\beta$ forces that $\dot{Q}_\beta$ forces that

$$\dot{G}(\beta) = \{ p(\beta) : p \in \dot{G} \}$$

is a filter in $\dot{P} = \dot{Q}_\beta$ meeting all $\dot{E}_\xi = \dot{D}_\xi$. This is preserved from $P_\beta$ to $P_{\omega_2}$ because $P_{\omega_2}$ is equivalent to $P_\beta^* P_{[\beta, \omega_2)}$.
Motivation: We will sketch the proof of the relative consistency (assuming the existence of a strongly inaccessible cardinal) of MA + \neg CH + There is no Kurepa tree
Proof.

Preparatory stage

1. First (using an inaccessible cardinal) obtain the consistency of CH + There is no Kurepa tree

2. And moreover for any c.c.c. forcing $P$ of cardinality $\omega_1$ $P\parallel -$ There is no Kurepa tree.

3. Assume: no c.c.c. forcing $P$ of cardinality $\omega_1$ forces that there is a Kurepa tree.
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2. Prove that if $P$ is c.c.c. and adds an uncountable branch through an $\omega_1$-tree, then there is $Q$ which is c.c.c., does not add uncountable branches through $\omega_1$-trees and

   $Q \models \neg \bar{P}$ is not c.c.c.
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$$Q \Vdash \check{P} \text{ is not c.c.c.}$$

3. Prove that if for each $\beta < \alpha$ we have $P_\beta \Vdash \check{Q}_\beta$ does not add an uncountable branches through $\omega_1$-trees, then $P_\alpha$ has this property as well as for each $\beta < \alpha$ we have that $P_\beta$ forces that $P_{[\beta, \alpha]}$ has this property.