On some problems concerning strong sequences

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Notation

If κ is an infinite cardinal and Q, R and S are collections of subsets of κ then the partition relation

$$Q \to (R, S)^n$$

holds iff for each $X \in Q$ and for each $f\colon [X]^n \to 2$ either $f([Y]^n) = \{0\}$ for some $Y \in R$ or $f([Z]^n) = \{1\}$ for some $Z \in S$. If $R = S$ then we write $Q \to (R)^n_2$.

Results for ultrafiltres

Ramsey (1930)
for non-principal ultrafiltres on ω

$$U \to (U)^2_2$$

Such ultrafiltres are called "Ramsey" (Galvin, around 1968)

Results for ultrafiltres

Baumgartner and Taylor (1978)

$$U \to (U, \omega)^2$$

and

$$U \to (U, 4)^3$$

Results for ultrafiltres

Sierpiński (1933)

$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Results for ideals

Duschnik and Miller (1941)
If κ is an infinite cardinal then

$$\kappa \to (\kappa, \omega)^2$$

Erdös - Rado (1956)
For κ - regular

$$\kappa \to (\kappa, \omega + 1)$$

Hajnal (1960)
If $2^{\aleph_0} = \aleph_1$ then

$$\omega_1 \not\to (\omega_1, \omega + 2)^2$$

Our goal

Theorem

For each $\lambda < \mu < \kappa$ such that κ, μ are regular numbers the following statement is true

$$\kappa \to (\mu)^2_\lambda$$

Strong sequences

Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p\colon p\colon T \to \{0,1\}\}.$$

For $s \subset T$, $i\colon s \to \{0,1\}$ it will be used the following notation

$$H_s^i = \{p \in D^T : p|s = i\}.$$

Efimov defined strong sequences in the subbase $\{H_{\{\alpha\}}^i : \alpha \in T\}$ of the Cantor cube and proved the following

Strong sequences

Theorem (Efimov)

Let κ be a regular, uncountable cardinal number. In the space D^T there is not a strong sequence

$$(\{H^i_{\{\alpha\}} : \alpha \in v_\xi\}, \{H^i_{\{\beta\}} : \beta \in w_\xi\}) \;\; ; \;\; \xi < \kappa$$

such that $|w_\xi| < \kappa$ and $|v_\xi| < \omega$ for each $\xi < \kappa$.

Strong sequences - Turzański results

Let X be a set, and $B \subset P(X)$ be a family of non-empty subsets of X closed with respect to finite intersections. Let S be a finite subfamily contained in B. A pair (S, H), where $H \subseteq B$, will be called *connected* if $S \cup H$ is centered.

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Definition (Turzański)

A sequence (S_ϕ, H_ϕ); $\phi < \alpha$ consisting of connected pairs is called *a strong sequence* if $S_\lambda \cup H_\phi$ is not centered whenever $\lambda > \phi$.

Strong sequences - Turzański results

Theorem (Turzański (1992))

If for $B \subset P(X)$ there exists a strong sequence $S = (S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ such that $|H_\phi| \leq \kappa$ for each $\phi < (\kappa^\lambda)^+$ then there exists a strong sequence $(S_\phi, T_\phi); \phi < \lambda^+$, where $|T_\phi| < \omega$ for each $\phi < \lambda^+$

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- If each of two elements in a set $A \subset X$ are compatible, then A is a *directed* set.
- A set A is κ- *directed* if every subset of X of cardinality less than κ has a bound, i.e. for each $B \subset X$ with $|B| < \kappa$ there exists $a \in A$ such that $(b, a) \in r$ for all $b \in B$.

Strong sequences on sets with relations

Definition

Let (X, r) be a set with relation r.
A sequence $(S_\phi, H_\phi); \phi < \alpha$ where $S_\phi, H_\phi \subset X$ and S_ϕ is finite is called a strong sequence if
1^o $S_\phi \cup H_\phi$ is ω-directed
2^o $S_\beta \cup H_\phi$ is not ω-directed for $\beta > \phi$.

Main results

Theorem

Let κ, μ where $\mu < \kappa$ be regular numbers. If there exists a strong sequences $(S_\alpha, H_\alpha)_{\alpha<\kappa}$ with $|H_\alpha| \leq \lambda$ for $\lambda < \kappa$, then there exists a strong sequence $(S_\alpha, T_\alpha)_{\alpha<\mu}$ with $|T_\alpha| < \omega$.

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Proof (compare: Protasov)
1) for $\lambda = \aleph_0$ we have Sierpiński theorem

$$2^{\aleph_0} \not\to (\aleph_1)^2_2.$$

2) for arbitrary λ.
Let us suppose that $2^{\lambda} \to (\lambda^{+})^2_2$. It means that for any partition

$$[2^{\lambda}]^2 = \bigcup\{A_{\alpha} : \alpha < \lambda\}$$

at least one A_{α} has cardinality λ^{+}.

Proof (cont.)
Let

$$2^{\lambda} = \{f\colon f\colon \lambda \to \{0,1\}\}$$

and let $f_1 \succeq f_2$ iff $f_1(\alpha) = 0$ and $f_2(\alpha) = 1$ for all $\alpha = \min\{\beta < \lambda\colon f_1(\beta) \neq f_2(\beta)\}$.
We can define

$$A_{\alpha} = \{\xi\colon \alpha = \min\{\beta < \lambda\colon f_{\xi}(\beta) \neq f_{\xi+1}(\beta)\}\}.$$

A_{α} contains only functions which form chain in the sense of $\succeq$ and let us consider the function $F\colon \lambda \to \lambda^{+}$ such that $F(\alpha) = \min A_{\alpha}$ for $A_{\alpha} \neq \emptyset$ and $F(\alpha) = 0$ for $A_{\alpha} = \emptyset$. Let us notice that $\sup\{F(\alpha)\colon \alpha < \lambda\} = \lambda^{+}$. But λ^{+} is regular. Contradiction.

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Corollary

If X is a Hausdorff space then

$$|X| \leq 2^{\chi(X)+c(X)}$$

Lemma

If for $A \subset P(X)$ there exists a strong sequence $(S_\alpha, H_\alpha)_{\alpha<(2^\lambda)^+}$ such that $|H_\alpha| \leq 2^\lambda$ for each $\alpha < (2^\lambda)^+$. then there exists a family $A \subset P(X)$ of cardinality λ^+ consisting of pairwise disjoint sets.

Sketch of the proof (corollary 4)
Let $\lambda = \chi(X) + c(X)$. Let us assume that $|X| > 2^{\lambda}$.
We can construct a sequence $\{x_{\alpha} \in X \colon \alpha < (2^{\lambda})^{+}\}$ and a strong sequence $(U_{\alpha}, \mathscr{B}_{\alpha})_{\alpha < (2^{\lambda})^{+}}$ with properties
1) U_{α}-open set such that $x_{\alpha} \in U_{\alpha}$
2) $\mathscr{B}_{\alpha}$ - local base in point x_{α}
3) $|\mathscr{B}_{\alpha}| \leq 2^{\lambda}$.
According to previous lemma we obtain a family consisting of pairwise disjoint sets of cardinality λ^{+}. Contradiction, because $\lambda \geq c(X)$.

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