Pseudocompact inverse primitive (semi)topological semigroups

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The Čech-Stone compactification of a Tychonoff space $X$ is a compact Hausdorff space $\beta X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space $Y$ extends to a continuous map $\bar{f} : \beta X \to Y$, i.e., the following diagram commutes:

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X & \xrightarrow{\beta} & \beta X \\
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Y & \xleftarrow{\beta} & \beta Y
\end{array}
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Every continuous map $f : X \to Y$ of Tychonoff spaces $X$ and $Y$ extends to the unique continuous map $\beta f : \beta X \to \beta Y$:

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The Glicksberg Theorem, 1959

For Tychonoff topological spaces $X$, $Y$ and $Z$ a continuous map $f : X \times Y \to Z$ extends to a continuous map $\overline{f} : \beta X \times \beta Y \to \beta Z$ if and only if the product $X \times Y$ is pseudocompact.

A Tychonoff topological spaces $X$ is called pseudocompact if every continuous map $f : X \to \mathbb{R}$ is bounded.

Reznichenko, 1994

Let $X$, $Y$ and $Z$ be Tychonoff topological spaces and continuous map $f : X \times Y \to Z$ be a continuous map. If $X$ are $Y$ pseudocompact then $f$ extends to a separately continuous map $\overline{f} : \beta X \times \beta Y \to \beta Z$. 
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**Definition**

Let $X$ and $Y$ be Tychonoff topological spaces. We shall say that $(X, Y)$ is a **Grothendieck pair** if every continuous image of $X$ in $C_p(Y)$ has the compact closure in $C_p(Y)$.

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If a Tychonoff pseudocompact space $X$ satisfies one of the following conditions: (i) $X$ is countably compact; (ii) $X$ has countable tightness; (iii) $X$ is separable; (iv) $X$ is a $k$-space, then $(X, Y)$ is a Grothendieck pair for every Tychonoff pseudocompact space $Y$. 
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Some Definitions

A topological space

- equipped with a continuous group operation and continuous inversion is called a *topological group*;
- equipped with a continuous group operation is called a *paratopological group*;
- equipped with a separately continuous group operation and continuous inversion is called a *quasitopological group*;
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A **semigroup** is a non-empty set with associative binary operation.

A semigroup \( S \)
- is called **inverse** if for every \( x \in S \) there exists unique \( y \in S \) such that \( xyx = x \) and \( yxy = y \), and in this case \( y \) is called inverse of \( x \) in \( S \) and denoted by \( x^{-1} \) (for an inverse semigroup \( S \) the map \( \iota : S \to S : x \mapsto x^{-1} \) is called inversion);
- is a **semilattice** if it is a commutative semigroup of idempotents;
- is **simple** if \( S \) does not contain proper ideals;
- is **0-simple** if \( S \) does not contains no proper ideals distinct from \( \{0\} \);
- is **completely simple** if \( S \) is simple and contains minimal left and right ideals;
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A subset idempotents $E(S)$ of a semigroup $S$ admits a natural partial order $\leq$:

$$e \leq f \quad \text{if and only if} \quad ef = fe = e, \quad e, f \in E(S).$$

An idempotent $e$ of a semigroup $S$ is primitive if it minimal in $E(S) \setminus \{0\}$.

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- equipped with a continuous semigroup operation and continuous inversion is called a topological inverse semigroup;
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Later we shall assume that all spaces are Hausdorff.
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We shall be interested in $\mathcal{C}$-compactifications for the following classes of semigroups:

- $\text{WAP}$ of compact semitopological semigroups;
- $\text{AP}$ of compact topological semigroups.

The corresponding $\mathcal{C}$-compactifications of a semitopological semigroup $S$ will be denoted by $\text{WAP}(S)$ and $\text{AP}(S)$. The notation came from the abbreviations for weakly almost periodic, almost periodic, and strongly almost periodic function rings that determine those compactifications.
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Semigroup compactifications

Reznichenko, 1995

For any Tychonoff countably compact semitopological semigroup $S$ the semigroup operation of $S$ extends to a separately continuous semigroup operation on $\beta S$, which implies that $\beta S$ coincides with the $WAP$-compactification of $S$.

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For any Tychonoff pseudocompact topological semigroup $S$ the semigroup operation of $S$ extends to a separately continuous semigroup operation $\beta S$, which implies that $\beta S$ coincides with the $WAP$-compactification of $S$.

Banakh, Dimitrova, 2010

For any Tychonoff topological semigroup $S$ with pseudocompact square $S \times S$ the semigroup operation of $S$ extends to a continuous semigroup operation on $\beta S$, which implies that $\beta S$ coincides with the $AP$-compactification of $S$. 
## The Comfort-Ross Theorem

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Theorem

For a Tychonoff space $X$ the following conditions are equivalent:

(i) $X$ is pseudocompact;

(ii) every locally finite family of non-empty open subsets of $X$ is finite;

(iii) every locally finite open cover of $X$ has a finite subcover.
Comfort-Ross like Theorem

**Ravsky, 2012**

The Tychonoff product of any non-empty family of pseudocompact paratopological groups is a pseudocompact space.

**G & Repovš, 2007**

Let $S$ be a 0-simple countable compact topological inverse semigroup. Then the Stone-Čech compactification of $S$ admits a structure of 0-simple topological inverse semigroup with respect to which the inclusion mapping of $S$ into $\beta S$ is a topological isomorphism.

**G & Pavlyk, 2013**

Let $\{S_i : i \in I\}$ be a non-empty family of primitive Hausdorff pseudocompact topological inverse semigroups. Then the direct product $\prod_{j \in I} S_j$ with the Tychonoff topology is a pseudocompact topological inverse semigroup.

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Let $\{S_i : i \in \mathcal{I}\}$ be a non-empty family of primitive Hausdorff pseudocompact topological inverse semigroups. Then the direct product $\prod_{j \in \mathcal{J}} S_j$ with the Tychonoff topology is a pseudocompact topological inverse semigroup.

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Let $S$ be a primitive pseudocompact topological inverse semigroup. Then the Stone-Čech compactification of $S$ admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of $S$ into $\beta S$ is a topological isomorphism.
Comfort-Ross like Theorem

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Let $S$ be a $0$-simple countable compact topological inverse semigroup. Then the Stone-Čech compactification of $S$ admits a structure of $0$-simple topological inverse semigroup with respect to which the inclusion mapping of $S$ into $\beta S$ is a topological isomorphism.

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Let $\{S_i : i \in J\}$ be a non-empty family of primitive Hausdorff pseudocompact topological inverse semigroups. Then the direct product $\prod_{j \in J} S_j$ with the Tychonoff topology is a pseudocompact topological inverse semigroup.

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Let $S$ be a primitive pseudocompact topological inverse semigroup. Then the Stone-Čech compactification of $S$ admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of $S$ into $\beta S$ is a topological isomorphism.
Let $S$ be a group and $\lambda$ be a cardinal $\geq 1$. On the set $B_\lambda(S) = (\lambda \times S \times \lambda) \sqcup \{0\}$ we define the semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $$(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0,$$ for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$.

The semigroup $B_\lambda(S)$ is called the Brandt semigroup. Every completely $0$-simple inverse semigroup is isomorphic to Brandt semigroup for some cardinal $\lambda$ and group $S$.

For all $\alpha, \beta \in \lambda$ we denote $S_{\alpha, \beta} = \{(\alpha, s, \beta) : s \in S\}$. 
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Let \( \{S_\iota : \iota \in \mathcal{I} \} \) be a disjoint family of semigroups with zero such that \( 0_\iota \) is zero in \( S_\iota \) for any \( \iota \in \mathcal{I} \). We put \( S = \{0\} \cup \bigcup \{S_\iota \setminus \{0_\iota\} : \iota \in \mathcal{I}\} \), where \( 0 \notin \bigcup \{S_\iota \setminus \{0_\iota\} : \iota \in \mathcal{I}\} \), and define a semigroup operation on \( S \) in the following way

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The semigroup \( S \) with such defined operation is called the orthogonal sum of the family of semigroups \( \{S_\iota : \iota \in \mathcal{I}\} \) and in this case we shall write \( S = \sum_{\iota \in \mathcal{I}} S_\iota \).

Petrich, 1984

A semigroup \( S \) is a primitive inverse semigroup if and only if \( S \) is the orthogonal sum of a non-empty family of Brandt semigroups.
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A semigroup \( S \) is a primitive inverse semigroup if and only if \( S \) is the orthogonal sum of a non-empty family of Brandt semigroups.
Let $S$ be a Hausdorff primitive inverse countably compact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in I\}$ of semitopological Brandt semigroups with zeros. Suppose that for every $i \in I$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}, \alpha_i \in \lambda_i$, such that at least one of the following conditions holds:

1. the group $(G_i)_{\alpha_i,\alpha_i}$ is left precompact;
2. $(G_i)_{\alpha_i,\alpha_i}$ is a pseudocompact paratopological group;
3. the group $(G_i)_{\alpha_i,\alpha_i}$ is left $\omega$-precompact pseudocompact;
4. the subsemigroup $S_{\alpha_i,\alpha_i} = (G_i)_{\alpha_i,\alpha_i} \cup \{0\}$ is a topological semigroup.

Then $S$ admits the unique topology which turns $S$ into a semitopological semigroup.

We recall that a group $G$ endowed with a topology is left $(\omega)$-precompact, if for each neighborhood $U$ of the unit of $G$ there exists a (countable) finite subset $F$ of $G$ such that $FU = G$. 

Oleg Gutik
Let $S$ be a semiregular primitive inverse pseudocompact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i): i \in \mathcal{I}\}$ of semitopological Brandt semigroups with zeros. Let for every $i \in \mathcal{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i \in \lambda_i$, such that at least the one of the following conditions holds:

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Let \( S \) be a Hausdorff primitive inverse pseudocompact topological semigroup and \( S \) be an orthogonal sum of the family \( \{B_{\lambda_i}(G_i) : i \in \mathcal{I}\} \) of topological Brandt semigroups with zeros. Then the following assertions hold:

\begin{enumerate}[(i)]
  \item every cardinal \( \lambda_i \) is finite;
  \item every maximal subgroup of \( S \) is open-and-closed subset of \( S \) and hence is pseudocompact;
  \item for every \( i \in \mathcal{I} \) the maximal Brandt semigroup \( B_{\lambda_i}(G_i) \) is a pseudocompact;
  \item if \( B(\alpha_i, e_i, \alpha_i) \) is a base of the topology at the unity \( (\alpha_i, e_i, \alpha_i) \) of a maximal non-zero subgroup \( (G_i)_{\alpha_i, \alpha_i} \) of \( S \), \( i \in \mathcal{I} \), such that \( U \subseteq (G_i)_{\alpha_i, \alpha_i} \) for any \( U \in B(\alpha_i, e_i, \alpha_i) \), then the family

\[ B(\beta_i, x, \gamma_i) = \left\{ (\beta_i, x, \alpha_i) \cdot U \cdot (\alpha_i, e_i, \gamma_i) : U \in B(\alpha_i, e_i, \alpha_i) \right\} \]

is a base of the topology at the point \( (\beta_i, x, \gamma_i) \in (G_i)_{\beta_i, \gamma_i} \subseteq B_{\lambda_i}(G_i) \), for all \( \beta_i, \gamma_i \in \lambda_i \);

\end{enumerate}

If in addition the topological space \( S \) is semiregular then

\begin{enumerate}[(v)]
  \item the family

\[ B_0 = \left\{ S \setminus \left( (G_{i_1})_{\alpha_{i_1}, \beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k}, \beta_{i_k}} \right) : i_1, \ldots, i_k \in \mathcal{I}, \alpha_{i_k}, \beta_{i_k} \in \lambda_{i_k}, \right. \]

\[ k \in \mathbb{N}, \left. \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})\} \right\} \text{ is finite} \}

is a base of the topology at zero of \( S \).
Let \( \{S_j : j \in J\} \) be a non-empty family of primitive semitopological inverse semigroups such that for each \( j \in J \) the semigroup \( S_j \) is either semiregular pseudocompact or Hausdorff countably compact, and moreover each maximal subgroup of \( S_j \) a pseudocompact paratopological group. Then the direct product \( \prod_{j \in J} S_j \) with the Tychonoff topology is a pseudocompact semitopological inverse semigroup.

Let \( \{S_i : i \in I\} \) be a non-empty family of primitive inverse semiregular pseudocompact (Hausdorff countably) topological semigroups. Then the direct product \( \prod_{j \in J} S_j \) with the Tychonoff topology is a pseudocompact inverse topological semigroup.
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Let $S$ be a primitive inverse pseudocompact quasi-regular topological semigroup. Then the Stone-Čech compactification of $S$ admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of $S$ into $\beta S$ is a topological isomorphism.

Let $S$ be a regular primitive inverse countably compact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in \mathcal{I}\}$ of semitopological Brandt semigroups with zeros. Suppose that for every $i \in \mathcal{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i \in \lambda_i$, such that at least one of the following conditions holds:

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Thank You for Your attention!