Ultrafilter selection properties

(joint work with Robert Bonnet and Stevo Todorčević)

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Definition

Let $\mathbb{B}$ be a Boolean algebra, $\kappa \leqslant |\mathbb{B}|$ an infinite cardinal.
We say that $\mathbb{B}$ has the **κ-selection property** if for every generating set $G \subseteq \mathbb{B}$ there exists an ultrafilter p on $\mathbb{B}$ such that

$$|p \cap G| \geqslant \kappa.$$

We say that $\mathbb{B}$ is **κ-Corson** if it fails the κ-selection property.

Definition

A Boolean algebra $\mathbb{B}$ has the **strong κ-selection property** if for every generating set $G \subseteq \mathbb{B}$ the set

$$\{p \in \mathrm{Ult}(\mathbb{B}) \colon |p \cap G| \geqslant \kappa\}$$

has nonempty interior.

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Fact

$\mathbb{B}$ *has the* $\aleph_0$*-selection property* $\implies$ $\mathbb{B}$ *is superatomic.*

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Definition

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Given a chain C, denote by $\mathbb{B}(C)$ the Boolean algebra generated by C.

Definition

A hereditarily interval algebra is a Boolean algebra whose all subalgebras are interval.

Open problem (R. Bonnet)

Find an uncountable hereditarily interval algebra.

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Theorem (folklore)

Every hereditarily interval algebra is of the form $\mathbb{B}(C)$, where $C \subseteq \mathbb{R}$.

Proof.

Assume $\mathbb{B} = \mathbb{B}(C)$, where C is a chain.

1. Neither ω_1 nor its inverse embed into C.
2. There is an uncountable set $G \subseteq C$ such that $|p \cap G| \leqslant \aleph_0$ for every $p \in \mathrm{Ult}(\mathbb{B})$.
3. $\mathbb{B}(G)$ is an uncountable interval algebra which fails the $\aleph_1$-selection property.
4. A contradiction (see one of the next slides).

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Theorem (Nikiel, Purisch, Treybig independently: Bonnet, Rubin)

$\mathbb{B}(\mathbb{R})$ *is not hereditarily interval.*

Definition

A poset algebra is a Boolean algebra $\mathbb{B}$ generated freely by a partially ordered set P. That is:

$$p_1 \wedge \ldots \wedge p_k \wedge \neg q_1 \wedge \ldots \wedge \neg q_\ell = 0 \implies (\exists\, i, j) \;\; p_i \leqslant q_j$$

for every $p_1, \ldots, p_k, q_1, \ldots, q_\ell$ in P. We write $\mathbb{B} = \mathbb{B}(P)$.

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Main results

Theorem

Let $\mathbb{B}$ be a poset Boolean algebra, let κ be a regular cardinal such that $\aleph_0 < \kappa \leqslant |\mathbb{B}|$. Then $\mathbb{B}$ has the κ-selection property.

Theorem

Let $\mathbb{B}$ be an interval Boolean algebra, $\aleph_0 < \lambda^+ < |\mathbb{B}|$. Then $\mathbb{B}$ has the strong λ^+-selection property.

Example

Let κ be any infinite cardinal. Then the free Boolean algebra with κ generators fails the strong $\aleph_0$-selection property.

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Preservation result

Theorem

Let $\mathbb{B}$ be a κ-Corson Boolean algebra, where $\kappa > \aleph_0$ is regular. Then every subalgebra of $\mathbb{B}$ is κ-Corson.

About the proofs

Definition

The pointwise topology τ_p on a Boolean algebra $\mathbb{B}$ is the topology generated by sets of the form

$$V_p^+ = \{a \in \mathbb{B} \colon a \in p\} \qquad \text{and} \qquad V_p^- = \{a \in \mathbb{B} \colon a \notin p\}$$

where $p \in \mathrm{Ult}(\mathbb{B})$.

Theorem

Let $\mathbb{B}$ be a κ-Corson Boolean algebra, where $\kappa = \operatorname{cf} \kappa > \aleph_0$. Then every open cover of $\langle \mathbb{B}, \tau_p \rangle$ contains a subcover of size $< \kappa$.

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Lemma

Let $\mathbb{B}$ be an infinite poset Boolean algebra. Then $\langle \mathbb{B}, \tau_p \rangle$ contains a closed discrete set of cardinality $|\mathbb{B}|$.

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Remark

Nakhmanson (1985) proved that the Lindelöf number of $C_p(K)$ is κ whenever K is a compact linearly ordered space of weight $\kappa \geqslant \aleph_0$.

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Elementary submodels

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Let $\theta > \kappa > \aleph_0$ be regular cardinals. An elementary submodel *M* of $\langle H(\theta), \in \rangle$ is **κ-stable** if $M \cap \kappa$ is an initial segment of κ.

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Given $A \in H(\theta)$ with $|A| < \kappa$, one can always find a κ-stable $M \preceq H(\theta)$ such that $A \in M$ and $|M| < \kappa$.

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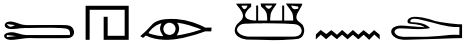
Crucial Lemma, going back to Bandlow ≈1990

Let $\mathbb{B}$ be a Boolean algebra, $\kappa = \operatorname{cf} \kappa > \aleph_0$. Then $\mathbb{B}$ is κ-Corson iff for every sufficiently closed κ-stable elementary submodel *M* of a big enough $H(\theta)$ there is a "canonical" projection

$$P_M \colon \mathbb{B} \to \mathbb{B} \cap M.$$



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