Ultrafilter selection properties

(joint work with Robert Bonnet and Stevo Todorcevic)

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Definition

Let $\mathbb{B}$ be a Boolean algebra, $\kappa \leq |\mathbb{B}|$ an infinite cardinal. We say that $\mathbb{B}$ has the $\kappa$-selection property if for every generating set $G \subseteq \mathbb{B}$ there exists an ultrafilter $p$ on $\mathbb{B}$ such that

$$|p \cap G| \geq \kappa.$$ 

We say that $\mathbb{B}$ is $\kappa$-Corson if it fails the $\kappa$-selection property.

Definition

A Boolean algebra $\mathbb{B}$ has the strong $\kappa$-selection property if for every generating set $G \subseteq \mathbb{B}$ the set

$$\{p \in \text{Ult}(\mathbb{B}) : |p \cap G| \geq \kappa\}$$

has nonempty interior.
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Fact

$\mathcal{B}$ has the $\aleph_0$-selection property $\implies$ $\mathcal{B}$ is superatomic.
Inspirations:

1. Banach space theory and Corson compact spaces
2. Interval Boolean algebras
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Definition
An interval Boolean algebra is a Boolean algebra generated by a linearly ordered set.
Given a chain $C$, denote by $\mathbb{B}(C)$ the Boolean algebra generated by $C$.

Definition
A hereditarily interval algebra is a Boolean algebra whose all subalgebras are interval.

Open problem (R. Bonnet)
Find an uncountable hereditarily interval algebra.
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Find an uncountable hereditarily interval algebra.
Theorem (folklore)

Every hereditarily interval algebra is of the form $\mathbb{B}(C)$, where $C \subseteq \mathbb{R}$.

Proof.

Assume $\mathbb{B} = \mathbb{B}(C)$, where $C$ is a chain.

1. Neither $\omega_1$ nor its inverse embed into $C$.
2. There is an uncountable set $G \subseteq C$ such that $|p \cap G| \leq \aleph_0$ for every $p \in \text{Ult}(\mathbb{B})$.
3. $\mathbb{B}(G)$ is an uncountable interval algebra which fails the $\aleph_1$-selection property.
4. A contradiction (see one of the next slides).
Theorem (folklore)

Every hereditarily interval algebra is of the form $B(C)$, where $C \subseteq \mathbb{R}$.

Proof.

Assume $B = B(C)$, where $C$ is a chain.

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Theorem (Nikiel, Purisch, Treybig independently: Bonnet, Rubin)

\( \mathcal{B}(\mathbb{R}) \) is not hereditarily interval.
Definition

A **poset algebra** is a Boolean algebra $\mathbb{B}$ generated freely by a partially ordered set $P$. That is:

$$p_1 \land \ldots \land p_k \land \neg q_1 \land \ldots \land \neg q_\ell = 0 \implies (\exists \ i, j) \ p_i \leq q_j$$

for every $p_1, \ldots, p_k, q_1, \ldots, q_\ell$ in $P$. We write $\mathbb{B} = \mathbb{B}(P)$.

Fact

*Every interval algebra is a poset algebra.*
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*Every interval algebra is a poset algebra.*
Main results

**Theorem**

Let $\mathbb{B}$ be a poset Boolean algebra, let $\kappa$ be a regular cardinal such that $\aleph_0 < \kappa \leq |\mathbb{B}|$. Then $\mathbb{B}$ has the $\kappa$-selection property.

**Theorem**

Let $\mathbb{B}$ be an interval Boolean algebra, $\aleph_0 < \lambda^+ < |\mathbb{B}|$. Then $\mathbb{B}$ has the strong $\lambda^+$-selection property.

**Example**

Let $\kappa$ be any infinite cardinal. Then the free Boolean algebra with $\kappa$ generators fails the strong $\aleph_0$-selection property.
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Example
Let $\kappa$ be any infinite cardinal. Then the free Boolean algebra with $\kappa$ generators fails the strong $\aleph_0$-selection property.
Theorem

Let $\mathcal{B}$ be a $\kappa$-Corson Boolean algebra, where $\kappa > \aleph_0$ is regular. Then every subalgebra of $\mathcal{B}$ is $\kappa$-Corson.
About the proofs

**Definition**

The pointwise topology $\tau_p$ on a Boolean algebra $\mathbb{B}$ is the topology generated by sets of the form

$$V_p^+ = \{ a \in \mathbb{B} : a \in p \} \quad \text{and} \quad V_p^- = \{ a \in \mathbb{B} : a \notin p \}$$

where $p \in \text{Ult}(\mathbb{B})$.

**Theorem**

Let $\mathbb{B}$ be a $\kappa$-Corson Boolean algebra, where $\kappa = \text{cf} \kappa > \aleph_0$. Then every open cover of $\langle \mathbb{B}, \tau_p \rangle$ contains a subcover of size $< \kappa$. 
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Lemma

Let $\mathbb{B}$ be an infinite poset Boolean algebra. Then $\langle \mathbb{B}, \tau_p \rangle$ contains a closed discrete set of cardinality $|\mathbb{B}|$.

Remark

Nakhmanson (1985) proved that the Lindelöf number of $C_p(K)$ is $\kappa$ whenever $K$ is a compact linearly ordered space of weight $\kappa \geq \aleph_0$. 

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W.Kubiš (http://www.math.cas.cz/kubis/)
Elementary submodels

Definition

Let $\theta > \kappa > \aleph_0$ be regular cardinals. An elementary submodel $M$ of $\langle H(\theta), \in \rangle$ is $\kappa$-stable if $M \cap \kappa$ is an initial segment of $\kappa$.

Fact

Given $A \in H(\theta)$ with $|A| < \kappa$, one can always find a $\kappa$-stable $M \preceq H(\theta)$ such that $A \in M$ and $|M| < \kappa$. 
Elementary submodels

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Let $\theta > \kappa > \aleph_0$ be regular cardinals. An elementary submodel $M$ of $\langle H(\theta), \in \rangle$ is $\kappa$-stable if $M \cap \kappa$ is an initial segment of $\kappa$.

**Fact**
Given $A \in H(\theta)$ with $|A| < \kappa$, one can always find a $\kappa$-stable $M \subseteq H(\theta)$ such that $A \in M$ and $|M| < \kappa$. 
Crucial Lemma, going back to Bandlow \( \approx 1990 \)

Let \( \mathcal{B} \) be a Boolean algebra, \( \kappa = \text{cf } \kappa > \aleph_0 \). Then \( \mathcal{B} \) is \( \kappa \)-Corson iff for every sufficiently closed \( \kappa \)-stable elementary submodel \( M \) of a big enough \( H(\theta) \) there is a “canonical” projection

\[
P_M : \mathcal{B} \to \mathcal{B} \cap M.
\]
THE END