Ideal versions of wQN-space and QN-space

Jaroslav Šupina

joint research with L. Bukovský and P. Das

Institute of Mathematics
Faculty of Science
P.J. Šafárik University in Košice

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A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called an ideal if

a) $B \in \mathcal{I}$ for any $B \subseteq A \in \mathcal{I}$,
b) $A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$,
c) $\mathrm{Fin} = [\omega]^{<\omega} \subseteq \mathcal{I}$,
d) $\omega \notin \mathcal{I}$.

$\mathcal{I}, \mathcal{J}$ are ideals in the following.

$$\mathcal{A} \subseteq \mathcal{P}(\omega) \qquad \mathcal{A}^d = \{A \subseteq \omega;\ \omega \setminus A \in \mathcal{A}\}$$

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a filter if $\mathcal{F}^d$ is ideal.

Convergence of reals $\langle x_n : n \in \omega \rangle$

H. Cartan [1937]

$$x' = \lim_{\boldsymbol{F}} f \qquad \text{if} \qquad f^{-1}(\boldsymbol{V}(x')) \subseteq \boldsymbol{F}$$

$f : \omega \to \mathbb{R} \qquad f(n) = x_n, n \in \omega$

$$x_n \xrightarrow{\mathcal{I}} x \quad \equiv \quad (\forall \varepsilon > 0)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |x_n - x| < \varepsilon)$$

All functions are assumed to be real-valued.

$\mathcal{I}$-convergence of $\langle f_n : n \in \omega \rangle$, $\quad f_n, f : X \to \mathbb{R}$

M. Katětov [1968], . . . , P. Kostyrko, T. Šalát and W. Wilczyński [2000]

$\mathcal{I}$-pointwise convergence $\qquad f_n \xrightarrow{\mathcal{I}} f$

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |f_n(x) - f(x)| < \varepsilon)$$

P. Das and D. Chandra [2013]

$\mathcal{I}$-quasinormal convergence $\qquad f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$

there exists $\langle \varepsilon_n : n \in \omega \rangle$ $\mathcal{I}$-converging to 0 such that

$$(\forall x \in X)(\exists A \in \mathcal{I})(\forall n \in \omega)(n \notin A \to |f_n(x) - f(x)| < \varepsilon_n)$$

M. Balcerzak, K. Dems and A. Komisarski [2007]

$\mathcal{I}$-uniform convergence $\qquad f_n \xrightarrow{\mathcal{I}\text{-u}} f$

$$(\forall \varepsilon > 0)(\exists A \in \mathcal{I})(\forall x \in X)(\forall n \in \omega)(n \notin A \to |f_n(x) - f(x)| < \varepsilon)$$

Á Császár and M. Laczkovich [1979], Z. Bukovská [1991]

Let $f_n, f, n \in \omega$ be functions on X. The following conditions are equivalent.

(i) $f_n \xrightarrow{\text{QN}} f$ on X.

(ii) There are sets $X_k \subseteq X$ such that $X = \bigcup_{k=0}^{\infty} X_k$ and $f_n \rightrightarrows f$ on X_k for every $k \in \omega$.

(iii) There are sets $X_k \subseteq X$ such that $X = \bigcup_{k=0}^{\infty} X_k$, $X_k \subseteq X_{k+1}, k \in \omega$ and $f_n \rightrightarrows f$ on X_k for every $k \in \omega$.

Moreover, if X is a topological space and $f_n, n \in \omega$ are continuous, then (i), (ii) and (iii) are equivalent to

(iv) There are closed sets $X_k \subseteq X$ such that $X = \bigcup_{k=0}^{\infty} X_k$, $X_k \subseteq X_{k+1}, k \in \omega$ and $f_n \rightrightarrows f$ on X_k for every $k \in \omega$.

$\mathcal{B} \subseteq \mathcal{I}$ is a base of $\mathcal{I}$ if for any $A \in \mathcal{I}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

$$\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{A}|;\ \mathcal{A} \subseteq \mathcal{I} \wedge \mathcal{A} \text{ is a base of } \mathcal{I}\}$$

Theorem

The following are equivalent:

a) $\operatorname{cof}(\mathcal{I}) = \kappa$.

b) *For any set X and for any sequence, if $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X there are $X_\xi, \xi < \kappa$ such that $X = \bigcup_{\xi<\kappa} X_\xi$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_ξ. Moreover, if X is a topological space and $f_n, n \in \omega$ are continuous, then the sets X_ξ can be chosen to be closed.*

R. Filipów and M. Staniszewski [2013] $\kappa = \aleph_0$

P. Das and D. Chandra [2013]

Let $f_n, f, n \in \omega$ be functions on X. If there are $X_k \subseteq X, k \in \omega$ such that $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_k then $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on $\bigcup_{k\in\omega} X_k$.

All spaces are assumed to be Hausdorff and infinite.

L. Bukovský, I. Recław and M. Repický [1991]

A topological space X is a QN-space (a wQN-space) if each sequence of continuous real-valued functions converging to zero on X is (has a subsequence) converging quasi-normally.

P. Das and D. Chandra [2013]

A topological space X is an $\mathcal{I}$QN-space (an $\mathcal{I}$wQN-space) if each sequence of continuous functions converging to zero on X is (has a subsequence) converging $\mathcal{I}$-quasinormally (with respect to its enumeration).

Ideals with a pseudounion

A set $B \subseteq \omega$ is called a pseudounion of the family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ if $\omega \setminus B$ is infinite and $A \subseteq^* B$ for any $A \in \mathcal{A}$.

Thus an ideal $\mathcal{I}$ is a P-ideal if and only if every countable subfamily of $\mathcal{I}$ has a pseudounion belonging to $\mathcal{I}$.

If a pseudounion A of $\mathcal{I}$ belongs to $\mathcal{I}$ then $\mathcal{I} = \{B \subseteq \omega;\ B \subseteq^* A\}$.

An ideal $\mathcal{I}$ has a pseudounion if and only if $\mathcal{I}$ is not tall.

If $\mathrm{cof}(\mathcal{I}) < \mathfrak{p}$ then $\mathcal{I}$ has a pseudounion.

$\emptyset \times \mathrm{Fin}$ has a pseudounion and $\mathrm{cof}(\emptyset \times \mathrm{Fin}) = \mathfrak{d}$.

$$\emptyset \times \mathrm{Fin} = \{A \subseteq \omega \times \omega;\ (\forall n \in \omega)\ \{m;\ (n,m) \in A\} \in \mathrm{Fin}\}$$

Ideals with a pseudounion

The n-th element of $A \subseteq \omega$ is denoted $\mathsf{e}_A(n)$.

Proposition

Let C be a pseudounion of an ideal $\mathcal{I}$, $A = \omega \setminus C$. Then

a) *For any sequence $\langle f_n : n \in \omega \rangle$ of real-valued functions on X, if $f_n \xrightarrow{\mathcal{I}} f$ then $f_{\mathsf{e}_A(n)} \to f$.*

b) *For any sequence $\langle f_n : n \in \omega \rangle$ of real-valued functions on X, if $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ then $f_{\mathsf{e}_A(n)} \xrightarrow{\mathrm{QN}} f$.*

Corollary

Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal with a pseudounion. Then

a) *Any topological space X is an $\mathcal{I}\mathrm{QN}$-space if and only if X is a QN-space.*

b) *Any topological space X is an $\mathcal{I}\mathrm{wQN}$-space if and only if X is a wQN-space.*

Non-increasing control

P. Kostyrko, T. Šalát and W. Wilczyński [2000]

The following are equivalent.

(i) $\mathcal{I}$ is a P-ideal.

(ii) For every sequence of reals $\{x_n\}_{n=0}^{\infty}$, if $x_n \xrightarrow{\mathcal{I}} x$ then there is $A \in \mathcal{I}^d$ such that $x_{\mathsf{e}_A(n)} \to x$.

R. Filipów and M. Staniszewski [2013]

The following are equivalent.

(i) $\mathcal{I}$ is a P-ideal.

(ii) For every sequence of functions $\langle f_n : n \in \omega \rangle$ on a set X, if $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ then there is a sequence of reals $\{\varepsilon_n\}_{n=0}^{\infty}$ converging to zero such that $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ with the control $\{\varepsilon_n\}_{n=0}^{\infty}$.

P. Das and D. Chandra [2013]

Let $\mathcal{I}$ be a P-ideal, $X = \bigcup\limits_{s \in S} X_s, |S| < \mathfrak{b}$.

$$\text{If } f_n \xrightarrow{\mathcal{I}\text{QN}} f \text{ on each } X_s \text{ then } f_n \xrightarrow{\mathcal{I}\text{QN}} f \text{ on } X.$$

If $\mathcal{I}$ is a P-ideal then $\quad \mathrm{add}(\mathcal{I}\text{QN-space}) \geq \mathfrak{b}$.

$\mathrm{add}(\mathcal{I}\text{QN-space}) = \min\{|\mathcal{A}|;\ (\forall A \in \mathcal{A})$ "A is an $\mathcal{I}$QN-space" $\wedge$ "$\bigcup \mathcal{A}$ is p.n. non-$\mathcal{I}$QN-space"$\}$

p.n.=perfectly normal

Archangel'skiĭ's property (α_1)

A.V. Arkhangel'skiĭ [1972]

A topological space Y is (α_1)-space if for any $\langle S_n : n \in \omega \rangle$ of sequences converging to some point $y \in Y$, there exists a sequence S converging to y such that $S_n \subseteq^* S$ for all $n \in \omega$.

A topological space Y is (α_1)-space if and only if for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to some point $y \in Y$, there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $\{x_{n,m};\ m \geq m_n, n \in \omega\}$ converges to y.

M. Scheepers [1998], L. Bukovský and J. Haleš [2007], M. Sakai [2007]

$\mathrm{C}_p(X)$ satisfies (α_1) if and only if X is a QN-space.

($\mathcal{I}$-α_1) For any continuous functions $f_{n,m} : X \to \mathbb{R}$ if $f_{n,m} \to 0$ for any $n \in \omega$ then there is a sequence $\langle B_n : n \in \omega\rangle$ of sets from $\mathcal{I}$ such that

$$(\forall \varepsilon > 0)(\forall x \in X)(\exists A \in \mathcal{I})(\forall n \in \omega)(m \notin A \cup B_n \to |f_{n,m}(x)| < \varepsilon).$$

Theorem

X satisfies ($\mathcal{I}$-α_1) if and only if X is an $\mathcal{I}$QN-space.

($\mathcal{I}$-α_1)

For any continuous functions $f_{n,m} : X \to \mathbb{R}$ if $f_{n,m} \to 0$ for any $n \in \omega$ then there is a sequence $\langle B_n : n \in \omega\rangle$ of sets from $\mathcal{I}$ such that

$$(\forall \varepsilon > 0)(\forall x \in X)(\exists A \in \mathcal{I})(\forall n \in \omega)(m \notin A \cup B_n \to |f_{n,m}(x)| < \varepsilon).$$

If $\mathrm{C}_p(X)$ satisfies ($\mathcal{I}$-α_1) then X is an $\mathcal{I}$QN-space.

1. $f_m \to 0 \qquad f_{n,m} = 2^n|f_m| \qquad f_{n,m} \to 0, n \in \omega,$
2. $\langle B_n : n \in \omega\rangle$, $B_n \subseteq B_{n+1}$, $\bigcup_{n=0}^{\infty} B_n = \omega$, $B_{-1} = \emptyset$
3. $m \in B_n \setminus B_{n-1} \qquad \varepsilon_m = 2^{-n},$
4. $\{m;\ \varepsilon_m \geq 2^{-n}\} = B_n \qquad \varepsilon_m \xrightarrow{\mathcal{I}} 0,$
5. $m \notin B_0 \qquad m \in B_n \setminus B_{n-1} \qquad \varepsilon_m = 2^{-n}$
6. $x \in X, \varepsilon = 1 \qquad m \notin A \cup B_n \qquad |f_m(x)| < \varepsilon_m.$

($\mathcal{I}$-α_1)

For any continuous functions $f_{n,m} : X \to \mathbb{R}$ if $f_{n,m} \to 0$ for any $n \in \omega$ then there is a sequence $\langle B_n : n \in \omega \rangle$ of sets from $\mathcal{I}$ such that

$$(\forall \varepsilon > 0)(\forall x \in X)(\exists A \in \mathcal{I})(\forall n \in \omega)(m \notin A \cup B_n \to |f_{n,m}(x)| < \varepsilon).$$

If X is an $\mathcal{I}$QN-space then $C_p(X)$ satisfies ($\mathcal{I}$-α_1).

1. $f_{n,m} \to 0, n \in \omega$ $\quad g_m = \sum_{n=0}^{\infty} \min\{2^{-n}, |f_{n,m}|\}$ $\quad g_m \to 0$, g_m continuous,
2. $g_m \xrightarrow{\mathcal{I}\mathrm{QN}} 0$ with the control $\varepsilon_m \xrightarrow{\mathcal{I}} 0$,
   - $x \in X$ $\quad A_x \in \mathcal{I}$ $\quad m \notin A_x \to g_m(x) < \varepsilon_m$,
   - $\langle B_n : n \in \omega \rangle$ $\quad m \notin B_n \to \varepsilon_m < 2^{-n}$,
3. $m \notin A_x \cup B_n$ $\quad g_m(x) < 2^{-n}$ $\quad |f_{n,m}(x)| < 2^{-n}$,
4. $x \in X$, $\varepsilon > 0$, $k_0 : 2^{-k_0} < \varepsilon$, m_0 $\quad (\forall k < k_0)(\forall m > m_0)\ |f_{k,m}(x)| < \varepsilon$.

L. Bukovský and J. Haleš [2007] (α_0), (α_0^*)

($\mathcal{I}$-α_0) For any continuous functions $f_{n,m}, f : X \to \mathbb{R}$ such that $f_{n,m} \to f$ for any $n \in \omega$ there is a sequence $\{n_m\}_{m=0}^{\infty}$ $\mathcal{I}$-divergent to ∞ such that $f_{n_m,m} \xrightarrow{\mathcal{I}} f$.

($\mathcal{I}$-α_0^*) For any continuous functions $f_{n,m}, f_n, f : X \to \mathbb{R}$ such that $f_{n,m} \to f_n$ for any $n \in \omega$ and $f_n \to f$ there is a sequence $\{n_m\}_{m=0}^{\infty}$ such that $f_{n_m,m} \xrightarrow{\mathcal{I}} f$.

Theorem

X satisfies $(\mathcal{I}\text{-}\alpha_0)$ $\equiv$ X satisfies $(\mathcal{I}\text{-}\alpha_0^*)$ $\equiv$ X is an $\mathcal{I}\mathrm{QN}$-space

Coverings, two parameters

L. Bukovský and J. Haleš [2007], M. Sakai [2007] $\alpha_1(\Gamma, \Gamma)$, β_1, β_2, β_3, β_1^*, β_2^*

A γ-cover $\langle U_n : n \in \omega \rangle$ is fully shrinkable if there is a closed γ-cover $\langle F_n : n \in \omega \rangle$ such that $F_n \subseteq U_n$ for each $n \in \omega$.

M. Sakai [2007]

Every open γ-cover $\langle U_n : n \in \omega \rangle$ of a perfectly normal space X is fully shrinkable if and only if X is a σ-set.

P. Das [2013]

A cover $\langle U_n : n \in \omega \rangle$ of X is an $\mathcal{I}$-γ-cover if $\{n \in \omega;\ x \notin U_n\} \in \mathcal{I}$ for each $x \in X$.

$$\mathcal{I}\text{-}\Gamma$$

An infinite cover $\mathcal{A}$ of X is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$. (J. Gerlits and Zs. Nagy [1982]) Γ

A topological space X is a σ-set if every F_σ subset of X is a G_δ set in X.

Theorem

The following are equivalent.

(i) *X is an $\mathcal{I}$QN-space.*

(ii) *For every sequence $\langle\{U_{n,m};\ m \in \omega\} : n \in \omega\rangle$ of fully shrinkable open γ-covers there is a sequence $\{n_m\}_{m=0}^{\infty}$ $\mathcal{I}$-divergent to ∞ such that $\{U_{n_m,m};\ m \in \omega\}$ is an $\mathcal{I}$-γ-cover of X.*

(iii) *For every sequence $\langle\{U_{n,m};\ m \in \omega\} : n \in \omega\rangle$ of fully shrinkable open γ-covers there is a sequence $\langle B_n : n \in \omega\rangle$ of sets from $\mathcal{I}$ such that*

$$(\forall x \in X)(\exists A \in \mathcal{I})(\forall n \in \omega)(m \notin A \cup B_n \to x \in U_{n,m}).$$

Proposition

Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be an ideal with a pseudounion C, $A = \omega \setminus C$. Then for any $\mathcal{I}$-γ-cover $\langle U_n : n \in \omega \rangle$, the sequence $\langle U_{e_A(n)} : n \in \omega \rangle$ is a γ-cover.

Corollary

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals with pseudounions. Then any topological space X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space if and only if X is an $\mathrm{S}_1(\Gamma, \Gamma)$-space.

Let $\mathcal{A}$, $\mathcal{B}$ be families of covers of X. A topological space X possesses the property $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$ if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers from $\mathcal{A}$ there exist sets $U_n \in \mathcal{U}_n$, $n \in \omega$ such that $\{U_n;\ n \in \omega\} \in \mathcal{B}$.
(M. Scheepers [1996])

P. Das and D. Chandra [2014]

A topological space X is an $(\mathcal{I},\mathcal{J})$wQN-space if each sequence of continuous functions $\mathcal{I}$-converging to zero on X has a subsequence converging $\mathcal{J}$-quasinormally (with respect to its enumeration).

Corollary

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals with pseudounions. Then any topological space X is an $(\mathcal{I},\mathcal{J})$wQN-space if and only if X is a wQN-space.

Proposition

Any γ-set is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$-space.

Proposition

If X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$-space then X is an $(\mathcal{I},\mathrm{Fin})\mathrm{wQN}$-space.

Corollary

Any γ-set is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space and an $(\mathcal{I},\mathcal{J})\mathrm{wQN}$-space.

A cover $\mathcal{A}$ of X is an ω-cover if for any finite subset F of X there is $A \in \mathcal{A}$ such that $F \subseteq A$. (J. Gerlits and Zs. Nagy [1982])

A topological space X is a γ-set if any open ω-cover of X contains γ-subcover. (J. Gerlits and Zs. Nagy [1982])

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Thanks for Your attention!