

A "mean" sequence in Boolean algebras

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Convergence

Definition

For $X \neq \emptyset$, a mapping $\lambda : X^\omega \rightarrow P(X)$ is a **convergence** on X .

A convergence λ is called a **topological convergence** iff there exists a topology \mathcal{O} on X such that $\lambda = \lim_{\mathcal{O}}$.

Theorem

Each topological convergence satisfy conditions:

$$(L1) \quad \forall a \in X \quad a \in \lambda(\langle a \rangle).$$

$$(L2) \quad \forall x \in X^\omega \quad \forall y \prec x \quad \lambda(x) \subset \lambda(y).$$

$$(L3) \quad \forall x \in X^\omega \quad (\forall y \prec x \quad \exists z \prec y \quad a \in \lambda(z)) \Rightarrow a \in \lambda(x).$$

If $|\lambda(x)| \leq 1$, then those are also sufficient conditions. (Kisyński, 1960)

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Generating topology

Theorem

For each convergence $\lambda : X^\omega \rightarrow P(X)$ there exists the maximal topology \mathcal{O}_λ such that $\forall x \in X^\omega \lambda(x) \subset \lim_{\mathcal{O}_\lambda} x$.

$$(L1) \quad \lambda'(x) = \begin{cases} \lambda(x) \cup \{a\} & \text{if } x = \langle a \rangle \text{ for some } a \in X \\ \lambda(x) & \text{otherwise.} \end{cases}$$

$$(L2) \quad \lambda'^-(x) = \bigcup_{x \prec y} \lambda(y)$$

$$(L3) \quad \lambda'^{-*}(x) = \bigcap_{y \prec x} \bigcup_{z \prec y} \lambda(z)$$

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$$\mathcal{O}_\lambda = \mathcal{O}_{\lambda'^{-*}}.$$

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The Cantor cube of weight κ , denoted by $\langle 2^\kappa, \tau_C \rangle$ is the Tychonov product of κ -many copies of two point discrete space $2 = \{0, 1\}$

Let $\xi : 2^\kappa \rightarrow P(\kappa)$ be a bijection defined by $f(x) = x^{-1}[\{1\}]$.

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For the sequence of sets $\langle X_n : n \in \omega \rangle \in (P(\kappa))^\omega$ let

$$\liminf_{n \in \omega} X_n = \bigcup_{k \in \omega} \bigcap_{n \geq k} X_n \quad \limsup_{n \in \omega} X_n = \bigcap_{k \in \omega} \bigcup_{n \geq k} X_n$$

Fact

A sequence $\langle x_n : n \in \omega \rangle$ converges to the point $x \in 2^\kappa$ iff

$$\liminf_{n \in \omega} X_n = \limsup_{n \in \omega} X_n = X,$$

where $X_n = \xi(x_n)$, and $X = \xi(x)$.

$\langle 2^\kappa, \tau_C \rangle$ is sequential iff $\kappa = \omega$.

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The sequential topology τ_s

Definition

$$\lambda_s(x) = \begin{cases} \{\limsup x\} & \text{if } \liminf x = \limsup x \\ \emptyset & \text{if } \liminf x < \limsup x \end{cases}$$

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λ_s satisfies (L1) i (L2), $|\lambda_s(x)| \leq 1$, so it is weakly topological, and since it must not satisfy (L3), $\lambda_s \neq \lim_{\tau_s}$.

Theorem

λ_s is a topological convergence iff \mathbb{B} is $(\omega, 2)$ -distributive.

$$a \in \lim_{\tau_s}(x) \Leftrightarrow \forall y \prec x \exists z \prec y a \in \lambda_s(z).$$

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Condition (\bar{h})

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A sequence x is lim sup-stable iff for each $y \prec x$ $\limsup y = \limsup x$.

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Complete Boolean algebra \mathbb{B} satisfies condition (\bar{h}) iff each sequence has a lim sup-stable subsequence.

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$\mathfrak{t}\text{-cc} \Rightarrow (\bar{h}) \Rightarrow \mathfrak{s}\text{-cc}$.

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Let x be a sequence in a c.B.a.

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λ_{ls} and λ_{li}

Definition

Let

$$\lambda_{ls}(x) = (\limsup x) \uparrow, \quad \lambda_{li}(x) = (\liminf x) \downarrow.$$

Theorem

Set $F \in \mathcal{F}_{ls}$ iff it is upward closed and $\bigwedge_{n \in \omega} x_n \in F$, for each decreasing $x \in F^\omega$.

Set $F \in \mathcal{F}_{li}$ iff it is downward closed and $\bigvee_{n \in \omega} x_n \in F$, for each increasing $x \in F^\omega$.

Open set in $\mathcal{O}_{\lambda_{ls}}$ is downward closed and contains $\mathbf{0}$.

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λ_{ls} and λ_{li} satisfy (L1) and (L2).

λ_{ls} and λ_{li} are topological convergences iff \mathbb{B} is $(\omega, 2)$ -distributive.

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Are λ_{ls}^* and λ_{li}^* topological convergences?

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Topology \mathcal{O}^*

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Let the family $\mathcal{P}^* = \mathcal{O}_{\lambda_{ls}} \cup \mathcal{O}_{\lambda_{li}}$ be a subbase for a topology \mathcal{O}^* .

Theorem

$$\mathcal{O}^* \subset \tau_s$$

$$\lim_{\mathcal{O}^*} = \lim_{\lambda_{ls}} \cap \lim_{\lambda_{li}}$$

Theorem

If \mathbb{B} satisfies (\hbar) or it is $(\omega, 2)$ -distributive, then $\lim_{\mathcal{O}^*} = \lim_{\tau_s}$.

Question 2.

Is it always $\lim_{\mathcal{O}^*} = \lim_{\tau_s}$?

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Is it always $\lim_{\mathcal{O}^*} = \lim_{\tau_s}$?

Topology \mathcal{O}^*

Definition

Let the family $\mathcal{P}^* = \mathcal{O}_{\lambda_{l_s}} \cup \mathcal{O}_{\lambda_{l_i}}$ be a subbase for a topology \mathcal{O}^* .

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Theorem

For Boolean algebra $P(\omega)$ we have $\mathcal{O}^* = \tau_s$.

Proof: Both spaces, $\langle P(\omega), \tau_s \rangle$ and $\langle P(\omega), \mathcal{O}^* \rangle$ are Hausdorff, $\mathcal{O}^* \subset \tau_s$ and $\langle P(\omega), \tau_s \rangle$ is homeomorphic to the Cantor cube, so it is compact, and as a compact space, it is minimal in the class of Hausdorff spaces, implying $\mathcal{O}^* = \tau_s$.

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Theorem

If a Boolean algebra carries strictly positive Maharam submeasure μ , we have $\mathcal{O}^* = \tau_s$.

Proof: For $O \in \tau_s$ and $a \in O$ let $B(a, r) = \{x \in \mathbb{B} : \mu(x \Delta a) < r\} \subset O$ and

$$O_1 = \{x \in \mathbb{B} : \mu(x \setminus a) < r/2\}, \quad O_2 = \{x \in \mathbb{B} : \mu(a \setminus x) < r/2\}$$

So, $A \in O_1 \cap O_2 \subset B(a, r) \subset O$.

Also we have $O_1 \in \mathcal{O}_{\lambda_{1s}}$ and $O_2 \in \mathcal{O}_{\lambda_{1i}}$.

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Does there exist a complete Boolean algebra such that $\mathcal{O}^* \neq \tau_s$?

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Definition

\mathcal{N}_0 is the family of all neighborhoods of the point 0 in topology τ_s .
 $\mathcal{N}_0^d = \{U \in \mathcal{N}_0 : U = U \downarrow\}$.

Theorem (Balcar, Glówczyński, Jech)

If $\langle \mathbb{B}, \tau_s \rangle$ is a Frechét space, then for each $V \in \mathcal{N}_0$ exists $U \in \mathcal{N}_0^d$ such that $U \subset V$. So, then \mathcal{N}_0^d is a neighborhood base at the point 0.

$\mathcal{N}_0^d = \mathcal{O}_{\lambda_{ls}}$ is a neighborhood base at the point 0 for the topology \mathcal{O}^* .

If in a topological space $\langle \mathbb{B}, \tau_s \rangle$ the family \mathcal{N}_0^d is not a neighborhood base at 0, then $\tau_s \neq \mathcal{O}^*$.

Question 4.

Does there exist a c.B.a. such that \mathcal{N}_0^d is not a neighborhood base of $\mathbf{0}$?

Base matrix tree

A base matrix tree is a tree $\langle \mathcal{T}, * \supset \rangle$ of height \mathfrak{h} such that \mathcal{T} is dense in a pre-order $\langle [\omega]^\omega, \subset^* \rangle$. Levels are MAD families, and maximal chains are towers.

By Balcar, Pelant and Simon, such tree always exists.

Let us denote by $Br(\mathcal{T})$ a set of all maximal branches of \mathcal{T} and let

$$\kappa = |Br(\mathcal{T})|.$$

$$2^{\omega_1} \leq \kappa \leq 2^{\mathfrak{c}}.$$

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The construction of a "mean" sequence

Let \mathbb{B} is a c.B.a. such that $\text{cc}(\mathbb{B}) > 2^c$ and $1 \Vdash_{\mathbb{B}} |((\mathfrak{h})^V)| = \check{\omega}$.

So, $1 \Vdash |\check{T}_\alpha| = \check{\omega}$

$1 \Vdash \exists X \in [\check{\omega}]^{\check{\omega}} \forall B \in \check{T}_\alpha X \subset B$

By Maximum principle there exists a name σ_α such that

$1 \Vdash \sigma_\alpha \in [\check{\omega}]^{\check{\omega}} \forall B \in \check{T}_\alpha X \subset B$

Let $\langle b_\alpha : \alpha < \kappa \rangle$ be a maximal antichain in \mathbb{B} . Then, by Mixing lemma there exists name τ such that

$$\forall \alpha < \kappa b_\alpha \Vdash \tau = \sigma_\alpha$$

$x_n = \|\check{n} \in \tau\|$ and for $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ we have

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Properties of "mean" sequence

$$0 \notin \lambda_{l_s}^*(x)$$

$$0 \in \lim_{\mathcal{O}_{\lambda_{l_s}}}(x)$$

Answer 1.

$\lambda_{l_s}^*$ is not a topological convergence.

$$0 \in \lim_{\mathcal{O}_{\lambda_{l_s}}}(x) \cap \lim_{\mathcal{O}_{\lambda_{l_i}}}(x) = \lim_{\mathcal{O}^*}(x) \text{ and } 0 \notin \lim_{\tau_s}(x)$$

Answer 2.

$$\lim_{\tau_s} \neq \lim_{\mathcal{O}^*}$$

Answer 3.

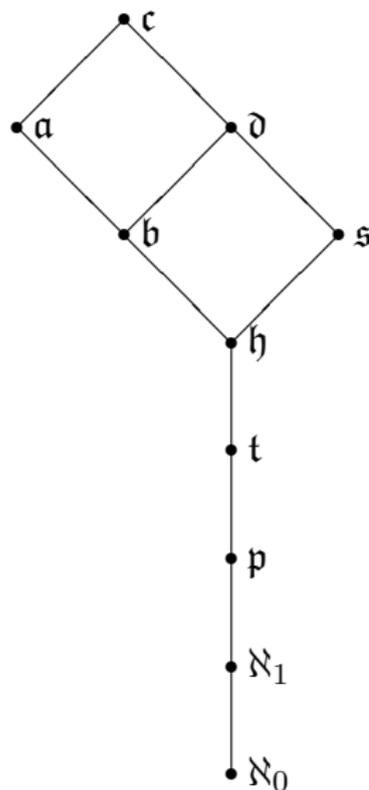
$$\tau_s \neq \mathcal{O}^*$$

If $X = \{x_n : n \in \omega\}$, then $\mathbb{B} \setminus X \in \tau_s$, but it is not downward closed and each downward closed neighborhood of 0 intersects X .

Answer 4.

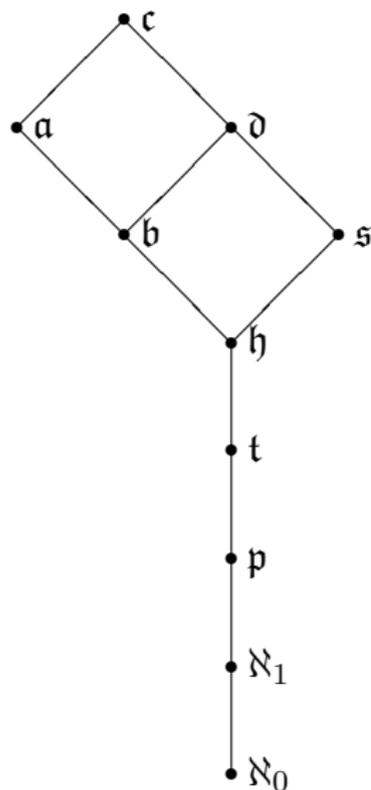
There exists a Boolean algebra in which \mathcal{N}_0^d is not a neighborhood base of 0 for topology τ_s .

Small cardinals


 \mathfrak{s}

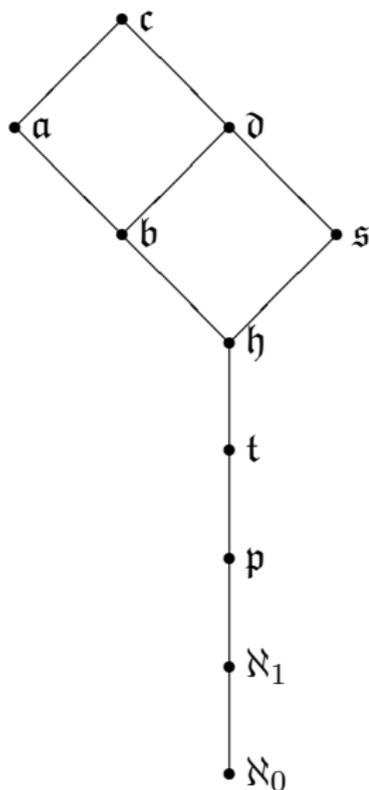
- Set $S \subset \omega$ splits a set $A \subset \omega$ iff $|A \cap S| = \omega$ and $|A \setminus S| = \omega$.
- $\mathcal{S} \subset [\omega]^\omega$ is a splitting family iff each $A \in [\omega]^\omega$ is splitted by some element of \mathcal{S} .
- Splitting number, \mathfrak{s} , is the minimal cardinality of a splitting family.

Small cardinals


 \mathfrak{t}

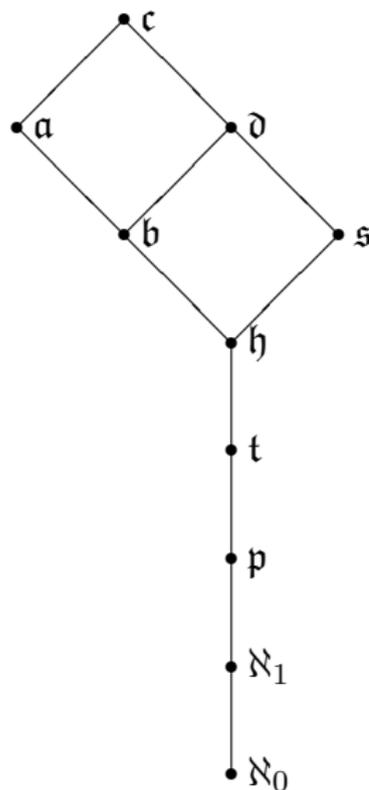
- $\mathcal{T} \subset [\omega]^\omega$ is a tower iff $\langle \mathcal{T}, * \supseteq \rangle$ well-ordered and the family \mathcal{T} has no pseudointersection.
- Tower number, \mathfrak{t} , is the minimal cardinality of a tower.

Small cardinals

**b**

- For functions $f, g \in \omega^\omega$, $f \leq^* g$ denotes $\exists n_0 \in \omega \forall n \geq n_0 f(n) \leq g(n)$.
- $\mathcal{B} \subset \omega^\omega$ is unbounded family iff there does not exist $g \in \omega^\omega$ such that $f \leq^* g$ for each $f \in \mathcal{B}$.
- Bounding number, \mathfrak{b} , is the minimal cardinality of unbounded family.

Small cardinals


 \mathfrak{h}

$$\mathfrak{h} = \min\{|\mathcal{H}| : \mathcal{H} \text{ is a family of open dense subsets of the order } \langle [\omega]^\omega, \subset^* \rangle \text{ and } \bigcap \mathcal{H} \text{ is not dense}\}.$$