Hjorth Analysis of General Polish Group Actions

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- A *Polish Topology* is a separable topology induced by a complete metric. A *Polish Space* is a topological space whose topology is polish.
- A subspace of a Polish space is Polish if and only it is G_δ.
- The product of a countable collection of Polish spaces is Polish. In particular, ω^ω and 2^ω are both Polish.

Polish Groups and Polish Actions

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- A continuous action of a Polish group G on a Polish space X is called a *Polish action*. We will denote by E_G^X the induced orbit equivalence relation on X.
- The orbit equivalence relation E_G^X is analytic, but not always Borel.

Logic Action

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- This is exactly the topology generated by

$$A_{\phi,\bar{a}} = \{\mathcal{M} \ : \ \mathcal{M} \models \phi(\bar{a})\}.$$

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- The induced orbit equivalence relation is $\simeq_{\mathcal{L}}$.

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- $(\mathcal{M}, \bar{a}) \equiv_{\alpha+1} (\mathcal{N}, \bar{b})$ if for every $c \in \omega$ there is $d \in \omega$ s.t. $(\mathcal{M}, \bar{a}^\frown c) \equiv_\alpha (\mathcal{N}, \bar{b}^\frown d)$ and for every $d \in \omega$ there is $c \in \omega$ s.t. $(\mathcal{N}, \bar{b}^\frown d) \equiv_\alpha (\mathcal{M}, \bar{a}^\frown c)$.

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- For λ limit, $(\mathcal{M}, \bar{a}) \equiv_\lambda (\mathcal{N}, \bar{b})$ if for every $\alpha < \lambda$, $(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{N}, \bar{b})$.

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- Given $\mathcal{M} \in Mod(\mathcal{L})$, there is $\alpha < \omega_1$ such that if $(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{M}, \bar{b})$ then $(\mathcal{M}, \bar{a}) \equiv_{\alpha+1} (\mathcal{M}, \bar{b})$.

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Definition

For $\mathcal{M} \in Mod(\mathcal{L})$, $\delta(\mathcal{M})$, the *Scott rank* of $\mathcal{M}$, is the least such α.

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  4. Given $\mathcal{M} \in Mod(\mathcal{L})$, for every $\mathcal{N} \in Mod(\mathcal{L})$:

     $$\mathcal{N} \equiv_{\delta(\mathcal{M})+\omega} \mathcal{M} \quad \Longrightarrow \mathcal{M} \simeq \mathcal{N}.$$

Scott Analysis

Theorem (Becker - Kechris)

$\simeq_{\mathcal{L}}$ is Borel if and only if there is an $\alpha < \omega_1$ such that for every $\mathcal{M} \in Mod(\mathcal{L})$, $\delta(\mathcal{M}) < \alpha$

Questions

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Generalize Scott analysis, or, find a topological version of Scott analysis.

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Question (Hjorth)

Let α be a countable ordinal. Is the following set Borel:

$$\mathbb{A}_\alpha = \{x \ : \ [x] \textit{ is } \mathbf{\Pi}^0_\beta \text{ for } \beta < \alpha + \omega\}$$

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Definition

$(x, U) \leq_\alpha (y, W)$ if and only if for every A a $\mathbf{\Pi}^0_\alpha$ set, if $W \Vdash g^* y \in A$ then $U \Vdash g^* x \in A$.

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3. $\leq_\alpha$ is Borel.

Hjorth Analysis

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Let x_0, x_1 in X, $\alpha < \omega_1$. $x_0 \equiv_\alpha x_1$ iff for all $V_1 \subseteq G$ nonempty and open there is $V_0 \subseteq G$ nonempty and open such that

$$(x_0, V_0) \leq_\alpha (x_1, V_1),$$

and vice versa:

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$\equiv_\alpha$ is a Borel and G - invariant equivalence relation.

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Suppose $A \subseteq X$ is an invariant $\mathbf{\Pi}^0_\alpha$ set, and $x \equiv_\alpha y$. Then $x \in A \iff y \in A$.

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Proof.

- Assume $x \in A$ for A a $\mathbf{\Pi}^0_\alpha$ invariant set.

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- As A is invariant, $G \Vdash g^* \cdot x \in A$.

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- Since $x \equiv_\alpha y$, there is a non empty and open W such that $(y, W) \leq_\alpha (x, G)$.

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- By the definition and the above, $W \Vdash g^* \cdot y \in A$. In particular, there is a g such that $g \cdot y \in A$.

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- By the definition and the above, $W \Vdash g^* \cdot y \in A$. In particular, there is a g such that $g \cdot y \in A$.
- By the invariance of A, y must be in A. □

So far...

1. A decreasing sequence $\equiv_\alpha$ of Borel equivalence relations which are invariant under G.
2. $E_G^X = \bigcap_{\alpha<\omega_1} \equiv_\alpha$.
3. A function $\delta : X \to (\omega_1, <)$ which is Borel and G - invariant.
4. There is an $\alpha < \omega_1$ such that for every $x \in X$ and for every $y \in X$:

$$x \equiv_{\delta(x)+\alpha} y \Longrightarrow \quad x \; E_G^X \; y.$$

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Proposition

Hjorth rank is G invariant and Borel. In fact:
For every countable ordinal α:

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is $\mathbf{\Pi}^0_{\alpha+k(\alpha)}$, for $k(\alpha) \in \omega$.

Scott's Isomorphism Theorem

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- The set $\{z \ : \ \delta(z) \leq \delta(x)\}$ is $\mathbf{\Pi}^0_{\delta(x)+m}$ for some $m \in \omega$.

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- So if $y \equiv_{\delta(x)+m} x$ then $\delta(y) \leq \delta(x)$.

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Proof.

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- So if $y \equiv_{\delta(x)+m} x$ then $\delta(y) \leq \delta(x)$.
- Hence if x and y are $\delta(x) + m + 1$ equivalent, they are orbit equivalent.

□

1st mission accomplished

1. A decreasing sequence $\equiv_\alpha$ of Borel equivalence relations which are invariant under G.
2. $E_G^X = \bigcap_{\alpha<\omega_1} \equiv_\alpha$.
3. A function $\delta : X \to (\omega_1, <)$ which is Borel and G - invariant.
4. There is an $\alpha < \omega_1$ such that for every $x \in X$ and for every $y \in X$:

$$x \equiv_{\delta(x)+\alpha} y \implies \quad x \; E_G^X \; y.$$

In our case, $\alpha = \omega$.

What about the boundedness principle ?

Theorem

E_G^X is Borel if and only if there is an α such that for every $x \in X$, $\delta(x) \leq \alpha$.

Complexity of $B \cdot x$

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- $F \cdot x$ is not necessarily Borel for F closed.

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- $G \cdot x$ is Borel.
- $F \cdot x$ is not necessarily Borel for F closed.

Proposition

$B \cdot x$ is Borel if and only if $B \cdot G_x$ is Borel. In particular, $U \cdot x$ is Borel, for U open.

Complexity of $B \cdot x$

Proposition

If $G \cdot x$ is $\mathbf{\Pi}^0_{\alpha+1}$ for $\alpha \geq 1$ then for every open U, $U \cdot x$ is $\mathbf{\Pi}^0_{\alpha+1}$.

Complexity of $B \cdot x$

Proposition

If $G \cdot x$ is $\mathbf{\Pi}^0_{\alpha+1}$ for $\alpha \geq 1$ then for every open U, $U \cdot x$ is $\mathbf{\Pi}^0_{\alpha+1}$.

Sketch of proof

- $\alpha = 1$: $G \cdot x$ is G_δ.

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Sketch of proof

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- Then $U \cdot x$ is open in $G \cdot x$, hence G_δ in X.

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Sketch of proof (ctd.)

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- We then apply a Theorem of Hjorth to refine the topology of X to a topology in which $G \cdot x$ is G_δ.
- Using the case $\alpha = 1$, $U \cdot x$ is G_δ in the new topology , and hence $U \cdot x$ was $\mathbf{\Pi}^0_{\alpha+1}$ in the original topology.

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- For all $U \subseteq G$ open, $U \cdot x$ is $\mathbf{\Pi}^0_{\alpha+1}$.
- It turns out that in this case, $\delta(x) \leq \alpha + 1$. □

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Proof.

$$A_\zeta = \{x \;:\; \delta(x) = \zeta\}$$

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Hjorth's question

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For α countable, the set

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Corollary

For every countable α, there are either countably many or perfectly many orbits that are $\mathbf{\Pi}^0_\beta$, for $\beta < \alpha + \omega$.