Minimally generated Boolean algebras and its Stone spaces

Miguel Angel Gaspar Arreola
mgaspar@matmor.unam.mx

Posgrado Conjunto en Ciencias Matemáticas
UNAM-UMSNH

Winter School, January 2014

In this talk we shall give some basic notions about minimally generated boolean algebras and some spaces related to this algebras.

This talk is motivated from the construction of a Efimov space from $\mathfrak{b} = \mathfrak{c}$ given by Dow and Shelah.

Definitions

Our algebras always will be subalgebras of $\mathcal{P}(\omega)$.

Definitions

Our algebras always will be subalgebras of $\mathcal{P}(\omega)$.

Definition

We say that $\mathbb{B}$ is a minimal extension of $\mathbb{A}$ ($\mathbb{A} <_m \mathbb{B}$), if there is no a proper subalgebra between them.

Definitions

Our algebras always will be subalgebras of $\mathcal{P}(\omega)$.

Definition

We say that $\mathbb{B}$ is a minimal extension of $\mathbb{A}$ ($\mathbb{A} <_m \mathbb{B}$), if there is no a proper subalgebra between them.

Definition

A boolean algebra $\mathbb{B}$ will be minimally generated if there is a sequence $\{a_\alpha : \alpha < \kappa\}$ such that $\mathbb{B}_\beta <_m \mathbb{B}_{\beta+1}$,

Definitions

Our algebras always will be subalgebras of $\mathcal{P}(\omega)$.

Definition

We say that $\mathbb{B}$ is a minimal extension of $\mathbb{A}$ ($\mathbb{A} <_m \mathbb{B}$), if there is no a proper subalgebra between them.

Definition

A boolean algebra $\mathbb{B}$ will be minimally generated if there is a sequence $\{a_\alpha : \alpha < \kappa\}$ such that $\mathbb{B}_\beta <_m \mathbb{B}_{\beta+1}$, where $\mathbb{B}_\beta = \langle\{a_\alpha : \alpha < \beta\}\rangle$.

Definitions

Our algebras always will be subalgebras of $\mathcal{P}(\omega)$.

Definition

We say that $\mathbb{B}$ is a minimal extension of $\mathbb{A}$ ($\mathbb{A} <_m \mathbb{B}$), if there is no a proper subalgebra between them.

Definition

A boolean algebra $\mathbb{B}$ will be minimally generated if there is a sequence $\{a_\alpha : \alpha < \kappa\}$ such that $\mathbb{B}_\beta <_m \mathbb{B}_{\beta+1}$, where $\mathbb{B}_\beta = \langle\{a_\alpha : \alpha < \beta\}\rangle$.
And it will be coherently minimally generated if for $\beta < \alpha$ $a_\alpha \setminus a_\beta \in \mathbb{B}_\alpha$.

Going back to the construction due to Dow and Shelah, they constructed a space that is the Stone space of a boolean algebra what is:

Going back to the construction due to Dow and Shelah, they constructed a space that is the Stone space of a boolean algebra what is:

- its definition is given by a game (actually, a strategy for the Scarborough-Stone game);

Going back to the construction due to Dow and Shelah, they constructed a space that is the Stone space of a boolean algebra what is:

- its definition is given by a game (actually, a strategy for the Scarborough-Stone game);
- mininmally generated;

Going back to the construction due to Dow and Shelah, they constructed a space that is the Stone space of a boolean algebra what is:

- its definition is given by a game (actually, a strategy for the Scarborough-Stone game);
- mininmally generated;
- each play of the game gives you a coherently minimally generated.

A topological space is sequentially compact if every infinite subset has a convergent subsequence.

A topological space is sequentially compact if every infinite subset has a convergent subsequence.

Proposition

If $\mathbb{B}$ is a coherently minimally generated boolean algebra, then its Stone space, $St(\mathbb{B})$, is sequentially compact.

A topological space is sequentially compact if every infinite subset has a convergent subsequence.

Proposition

If $\mathbb{B}$ is a coherently minimally generated boolean algebra, then its Stone space, $St(\mathbb{B})$, is sequentially compact.

Corollary

If $\mathbb{B}$ is a coherently minimally generated boolean algebra then $St(\mathbb{B}) \setminus \{p\}$ is sequentially compact if and only if there is no sequence from $St(\mathbb{B})$ converging to p.

The Scarborough-Stone Game

In this game the players are going to construct a coherently minimally generated subalgebra of $\mathcal{P}(\omega)$ via a sequence of the generators.

The Scarborough-Stone Game

In this game the players are going to construct a coherently minimally generated subalgebra of $\mathcal{P}(\omega)$ via a sequence of the generators.

For the first ω steps just take $a_n = \omega \setminus n$, and then the interesting part happens...

For $\alpha \geq \omega$:

For $\alpha \geq \omega$:

- Player I plays a partition $\{b_\alpha, \omega \setminus b_\alpha\}$;

For $\alpha \geq \omega$:

- Player I plays a partition $\{b_\alpha, \omega \setminus b_\alpha\}$;
- Player II chooses an element of such partition, let say a_α.

For $\alpha \geq \omega$:

- Player I plays a partition $\{b_\alpha, \omega \setminus b_\alpha\}$;
- Player II chooses an element of such partition, let say a_α.

Player I wins at stage λ if the space $St(\mathbb{B}_\lambda) \setminus \{p_\lambda\}$ is sequentially compact,

For $\alpha \geq \omega$:

- Player I plays a partition $\{b_\alpha, \omega \setminus b_\alpha\}$;
- Player II chooses an element of such partition, let say a_α.

Player I wins at stage λ if the space $St(\mathbb{B}_\lambda) \setminus \{p_\lambda\}$ is sequentially compact, (where p_λ is the filter generated by the sequence of generators).

For $\alpha \geq \omega$:

- Player I plays a partition $\{b_\alpha, \omega \setminus b_\alpha\}$;
- Player II chooses an element of such partition, let say a_α.

Player I wins at stage λ if the space $St(\mathbb{B}_\lambda) \setminus \{p_\lambda\}$ is sequentially compact, (where p_λ is the filter generated by the sequence of generators).

At stage λ if Player I has not already won, then Player II chooses a sequence of points from $St(\mathbb{B}_\lambda)$ that converges to p_λ, and Player II wins at stage $\lambda + 1$ if such sequence still converges to $p_{\lambda+1}$.

Remark

If Player I has a winning strategy, then it gives you an Efimov space and if Player II does not have a winning strategy then it gives you a Scarborough-Stone example.

Remark

If Player I has a winning strategy, then it gives you an Efimov space and if Player II does not have a winning strategy then it gives you a Scarborough-Stone example.

A winning strategy for Player I in this game is what is expected to be.

Remark

If Player I has a winning strategy, then it gives you an Efimov space and if Player II does not have a winning strategy then it gives you a Scarborough-Stone example.

A winning strategy for Player I in this game is what is expected to be.

I like more the topological version of that, it reduces to achieve weak-D property in each step.

Why is that?

Why is that?

- The closed discrete sets in $St(\mathbb{B}) \setminus \{p\}$ are exactly the converging sequences to p.

Why is that?

- The closed discrete sets in $St(\mathbb{B}) \setminus \{p\}$ are exactly the converging sequences to p.
- To kill the sequence we have to blow up it to locally finite family of clopen sets.

Why is that?

- The closed discrete sets in $St(\mathbb{B}) \setminus \{p\}$ are exactly the converging sequences to p.
- To kill the sequence we have to blow up it to locally finite family of clopen sets.
- Here is where it is crucial the weak-D property because it is easy to see that the extension of the algebra is minimal if and only if the corresponding space has the weak-D property.

We are interested in that kind of spaces. One of the most clear spaces that we can get in that way is taking a tower $\{F_\alpha : \alpha < \mathfrak{t}\}$ and set $a_{\omega+\alpha}$ to be F_α.

We are interested in that kind of spaces. One of the most clear spaces that we can get in that way is taking a tower $\{F_\alpha : \alpha < \mathfrak{t}\}$ and set $a_{\omega+\alpha}$ to be F_α.

It is easy to see that it gives you a coherently minimally generated boolean algebra and its stone space without a point is sequentially compact.

We are interested in that kind of spaces. One of the most clear spaces that we can get in that way is taking a tower $\{F_\alpha : \alpha < \mathfrak{t}\}$ and set $a_{\omega+\alpha}$ to be F_α.

It is easy to see that it gives you a coherently minimally generated boolean algebra and its stone space without a point is sequentially compact.

In fact, Nyikos and Vaughan proved that those are exactly the Franklin-Rajagopalan spaces.

Let $\mathcal{F}$ be an ultrafilter, we want to find $\{a_\alpha : \alpha < \kappa\} \subset \mathcal{F}$ such that its related space have nice properties, sequentially compactness is de golden property.

Let $\mathcal{F}$ be an ultrafilter, we want to find $\{a_\alpha : \alpha < \kappa\} \subset \mathcal{F}$ such that its related space have nice properties, sequentially compactness is de golden property.

An easy case happens if the ultrafilter contains a tower, as seen before.

Let $\mathcal{F}$ be an ultrafilter, we want to find $\{a_\alpha : \alpha < \kappa\} \subset \mathcal{F}$ such that its related space have nice properties, sequentially compactness is de golden property.

An easy case happens if the ultrafilter contains a tower, as seen before.

If each a_α were a pseudointersection of the family of the generators that appear before it,

Let $\mathcal{F}$ be an ultrafilter, we want to find $\{a_\alpha : \alpha < \kappa\} \subset \mathcal{F}$ such that its related space have nice properties, sequentially compactness is de golden property.

An easy case happens if the ultrafilter contains a tower, as seen before.

If each a_α were a pseudointersection of the family of the generators that appear before it, then it fails in the D-property when the previous case does not happen.

Then we can kill every sequence of natural numbers but it is not enough, it could be enough if we ensure to carry the D property along the construction.

Then we can kill every sequence of natural numbers but it is not enough, it could be enough if we ensure to carry the D property along the construction.

For that is enough to preserve D property while killing pseudointersections.

Then we have a method and we have a plan, should it be easy right?

Then we have a method and we have a plan, should it be easy right?

Not at all.

Thank you!

THE END