Porous sets and Martin Numbers

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Definition

Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$ is σ-*centered* if there is a family $\{P_i : i \in \omega\}$ of subsets of $\mathbb{P}$ such that for every k and every $p, q \in P_k$ there is $p' \in P_k$ such that $p' \leq p$ and $p \leq q$.

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Definition

Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$ is *σ-n-linked* if there is a family $\{P_i : i \in \omega\}$ of subsets of $\mathbb{P}$ such that for every k and every sequence $\{p_i : i < n\} \subseteq P_k$ there is $p \in \mathbb{P}$ such that $p \leq p_i$ for every $i < n$.

Definition

Given a cardinal κ and a property φ about partial orders, we define $MA_{\varphi}(\kappa)$ as the following statement:
For every partial order such that $\varphi(\mathbb{P})$ and every family of dense sets $\mathcal{D}$ of $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$, there is a filter $F \subseteq \mathbb{P}$ such that F intersects every element of $\mathcal{D}$.

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- MA is the statement $\mathfrak{m}_{c.c.c.} = \mathfrak{c}$.
- $\mathfrak{m}_{\sigma\text{-}centered} = \mathfrak{t}$

Proposition

$\mathfrak{m}_{\textit{c.c.c.}} \leq \mathfrak{m}_{\sigma\textit{-linked}} \leq \mathfrak{m}_{\sigma\text{-}3\textit{-linked}} \leq \cdots \leq \mathfrak{m}_{\sigma\textit{-centered}}$

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I'll try to construct a model where an infinite amount of these cardinal invariants are diferent.

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A subset $A \subseteq 2^\omega$ is porous of degree n if for every $s \in 2^{<\omega}$ there is $t \in 2^n$ such that $\langle s \frown t \rangle \cap A = \emptyset$. A subset $A \subseteq 2^\omega$ is porous if it is porous of degree n for some $n \in \omega$.

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There are canonical porous sets: for every $\sigma : 2^{<\omega} \to 2^n$ let

$$X_\sigma = \{x \in 2^\omega : \forall k \in \omega (x \notin \langle x|k \frown \sigma(x|k) \rangle)\}.$$

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As a consequence we can use functions to code porous sets.

For every n consider $\mathbf{SP}_n$ the σ-ideal generated by porous sets of degree n.

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It is easy to see that $Ctbl = \mathbf{SP}_1 \subsetneq \mathbf{SP}_2 \subsetneq \mathbf{SP}_3 \subsetneq \cdots \subseteq \mathbf{SP}$.
Therefore $\omega_1 \leq \text{non}(\mathbf{SP}_2) \leq \text{non}(\mathbf{SP}_3) \leq \ldots \leq \text{non}(\mathbf{SP})$.

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Definition

We say that a partial order $\mathbb{P}$ strongly preserves $\text{non}(\mathbf{SP}_n)$ if for every name $\dot{X}$ for a porous set of degree n such that $\Vdash_P$ "$\dot{X} \subseteq \check{2}^\omega$", there is $Y \in \mathbf{SP}_n$ such that $\Vdash_P$ "$\dot{X} \subseteq Y$"

Lemma

Finite support iteration of c.c.c. forcings that strongly preserves $\mathrm{non}(\mathbf{SP}_n)$ *strongly preserves* $\mathrm{non}(\mathbf{SP}_n)$.

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Lemma

If $\mathbb{P}$ *is a* σ*-*2^n*-linked forcing, then* $\mathbb{P}$ *strongly preserves* $\mathrm{non}(\mathbf{SP}_n)$.

$\mathbb{P}^0_n(X) = \{\langle s, F\rangle :$ **(a)** s is a finite partial function from $2^{<\omega}$ to 2^n,
(b) $F \in [X]^{<\omega}$,
(c) for each $\sigma \in \mathrm{dom}(s)$, $F \cap \langle \sigma^\frown s(\sigma)\rangle = \emptyset$,
(d) F is a strongly porous set of degree $n\}$

and define $\langle s, F\rangle \leq \langle s', F'\rangle$ if $s' \subseteq s$ and $F' \subseteq F$.

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and define $\langle s, F\rangle \leq \langle s', F'\rangle$ if $s' \subseteq s$ and $F' \subseteq F$.

Proposition

Let $\mathbb{P}_n(X) = (\mathbb{P}^0_n(X))^{<\omega}$. *Then* $\Vdash_{\mathbb{P}}$ "$\check{X} \in \mathbf{SP}_n$".

Proposition

$\mathbb{P}_n(X)$ is a σ-$2^n - 1$-linked forcing. As a consequence $\mathbb{P}_n(X)$ strongly preserves $\mathrm{non}(\mathbf{SP}_n)$.

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$\mathfrak{m}_{\sigma\text{-}2^n-1\text{-}linked} \leq \mathrm{non}(\mathbf{SP}_n)$

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$\mathbb{P}_n(X)$ is a σ-2^n-1-linked forcing. As a consequence $\mathbb{P}_n(X)$ strongly preserves $\mathrm{non}(\mathbf{SP}_n)$.

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Theorem

It is consistent with ZFC that $\mathrm{non}(\mathbf{SP}_n) < \mathrm{non}(\mathbf{SP}_{n+1})$.

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What about the rest of the cardinal invariants associated?

Theorem

There is a model of ZFC such that for every $n \in \omega$, $\mathrm{non}(\mathbf{SP}_{n+1}) = \mathfrak{m}_{2^n} = \omega_{n+1}$.

What about the rest of the cardinal invariants associated?

$$\mathrm{add}(\mathbf{SP}) = min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{SP} \wedge \bigcup \mathcal{A} \notin \mathbf{SP}\}$$

$$\mathrm{cov}(\mathbf{SP}) = min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{SP} \wedge \bigcup \mathcal{A} = 2^{\omega}\}$$

$$\mathrm{cof}(\mathbf{SP}) = min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbf{SP} \wedge \mathcal{A} \text{ is a cofinal family }\}$$

- about add(**SP**)

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It's easy to see that
$\text{cov}(\mathbf{SP}) \leq \cdots \leq \text{cov}(\mathbf{SP}_3) \leq \text{cov}(\mathbf{SP}_2) \leq \text{cov}(\mathbf{SP}_1) = \mathfrak{c}$

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Theorem (Hrušák, Zindulka)

It is consistent with ZFC that $\mathrm{cof}(\mathbf{N}) < \mathrm{cov}(\mathbf{SP})$.