Extremal disconnectedness and ultrafilters

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Contents

Extremally disconnected spaces

Definition (Stone, 1937)

A topological space is called *extremally disconnected* (or *ED* for short) if it is regular and the closure of every open set is open, or equivalently, the closures of any two disjoint open sets are disjoint.

- Every ED space is zero-dimensional and Tychonoff.
- Every open (or dense) subspace of an ED space is also an ED space.
- Every discrete space is ED, but the converse is not true (*e.g.*, $\beta\omega$).
- Every metrizable ED space is discrete.

Extremal disconnectedness can be considered as a **non-trivial generalization of discreteness**. This notion has been studied by many authors for several years.

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Arhangel'skii's problem

Problem (Arhangel'skii, 1967)

Is there a nondiscrete extremally disconnected topological group?

Contents

Consistent examples

Partial positive solutions

For each one of the following assumptions, there is an example answering Arhangel'skii's question:

- (Sirota, 1969/Louveau, 1972) There is a Ramsey ultrafilter on ω.
- (Malykhin,1975) $\mathfrak{p} = \mathfrak{c}$.

Malykhin's construction was based on Hindman's Theorem (a very useful result of Ramsey theory). His construction gives an union ultrafilter. These group topologies are on the countable Boolean group $([\omega]^{<\omega}, \Delta)$. In fact, Arhangel'skii's question reduces to the Boolean case.

Theorem (Malykhin, 1975)

Any ED topological group must contain an open (and therefore closed) Boolean subgroup (i.e., a subgroup consisting of elements of order 2).

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The classical consistent example

Given a filter $\mathcal{F}$ on ω, $\mathcal{F}^{<\omega} = \{[F]^{<\omega} \colon F \in \mathcal{F}\}$ induces a group topology $\tau_{\mathcal{F}}$ on the Boolean group $([\omega]^{<\omega}, \Delta)$ by declaring $\mathcal{F}^{<\omega}$ to be the filter of neighbourhoods of the $\emptyset$.

Theorem (Louveau, 1972)

The group $([\omega]^{<\omega}, \tau_{\mathcal{F}})$ is ED if and only if $\mathcal{F}$ is a Ramsey ultrafilter.

The same construction works on a measurable cardinal, and yet another example can be obtained from Matet forcing with a union-ultrafilter.

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The spectrum of a topological space

Definition

Let (X, τ) be a topological space and $x \in X$. The *spectrum* of X at x, denoted by $\mathrm{Spect}(X, x)$, is defined as the collection of all $p \in \omega^*$ such that there is a sequence $\langle U_n \colon n \in \omega \rangle \subset \tau$ with $p = \{A \subset \omega \colon x \in \overline{\bigcup_{n \in A} U_n}\}$.

The *spectrum* of X is $\mathrm{Spect}(X) = \bigcup_{x \in X} \mathrm{Spect}(X, x)$.

- 0-$\mathrm{Spect}(X)$, when $\langle U_n \colon n \in \omega \rangle \subset CO(X)$.
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Some facts about Spect(X)

When X is ED, Spect(X) is more than not empty.

Proposition

Let X be an ED space. Then

1. 0-C-Spect(X) = 0-Spect(X) = Spect(X).
2. If $\chi(X, x) < \partial$, then Spect(X) $\subset$ P-points.
3. If $p \in$ Spect(X) and $q \leqslant_{RK} p$, then $q \in$ Spect(X).
4. For every $p, q \in$ Spect(X) there is $r \in$ Spect(X) such that $p, q \leqslant_{RK} r$.

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Let $\mathbb{G}$ be an ED topological group and $p, q \in \text{Spect}(X)$. Is there $r \in \text{Spect}(X)$ such that $r \leqslant_{RK} p, q$?

Consistently, yes. The NCF principle implies that for every $p, q \in \omega^*$ there is $r \in \omega^*$ such that $r \leqslant_{RB} p, q$.

Question

What kind of ultrafilters can live in Spect(X)?

Ramsey ultrafilters, P-points, nwd-ultrafilters. . . but what about ED topological groups? The spectrum of the consistent examples for Arhangel'skii's question that we know contain at least one P-point.

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Not adding Cohen reals

Given a topological space X, let $RO(X)$ be the algebra of regular open sets of X. If X is an ED space, then $RO(X) = CO(X)$.

Proposition

Let X be an ED space. Then $RO(X)$ does not add Cohen reals if and only if for every continuous function $f : X \to 2^\omega$ there exists a non-empty open set U such that $f[U] \in \text{nwd}(2^\omega)$.

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Definition (Baumgartner, 1995)

An ultrafilter p on ω is *nowhere dense* (nwd) if for any $f : \omega \to 2^\omega$ there is $A \in p$ such that $f[A] \in \mathrm{nwd}(2^\omega)$.

- (Baumgartner, 1995) Every P-point is a nwd-ultrafilter.

Theorem (Błaszczyk-Shelah, 2001)

The following are equivalent.

- *There is a* nwd-*ultrafilter on* ω.
- *There is a non-trivial* σ*-centered forcing notion which does not add Cohen reals.*

Theorem (Shelah,1998)

It is consistent with ZFC *that there is no* nwd-*ultrafilter on* ω.

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An ultrafilter p on ω is *nowhere dense* (nwd) if for any $f : \omega \to 2^\omega$ there is $A \in p$ such that $f[A] \in \text{nwd}(2^\omega)$.

- (Baumgartner, 1995) Every P-point is a nwd-ultrafilter.

Theorem (Błaszczyk-Shelah, 2001)

The following are equivalent.

- *There is a* nwd-*ultrafilter on* ω.
- *There is a non-trivial* σ*-centered forcing notion which does not add Cohen reals.*

Theorem (Shelah,1998)

It is consistent with ZFC *that there is no* nwd-*ultrafilter on* ω.

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The consistent examples for Arhangel'skii's question that we know satisfy the following property.

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For every continuous function $f : \mathbb{G} \to 2^\omega$ there exists a non-empty open set U such that $f[U] \in \mathrm{nwd}(2^\omega)$.

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For every ED topological group $\mathbb{G}$ and for every continuous function $f : \mathbb{G} \to 2^\omega$ there is an non-empty open set U such that $f[U] \in \mathrm{nwd}(2^\omega)$.

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Theorem

Let $\mathbb{G}$ be an ED countable Boolean topological group. If $f : \mathbb{G} \to 2^\omega$ is a continuous homomorphism, then there is a non-empty open set U such that $f[U] \in \mathrm{nwd}(2^\omega)$.

If Hrušák's conjecture is true, then the existence of a nondiscrete separable ED topological group implies the existence of a nwd-ultrafilter and thus, the existence of a nondiscrete separable ED topological group will be independent of ZFC.

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