Simplest Possible Wellorders of $H(\kappa^+)$

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*presenting joint work with Philipp Lücke*

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Question

How simple a wellordering of $H(\kappa^+)$ can one have definably (by a first order formula in the language of set theory) over $H(\kappa^+)$?
Basic Motivation

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We want to measure complexity in terms of the standard Lévy hierarchy and in terms of the necessary parameters. Note that definable wellorders of $H(\omega_1)$ are closely connected to definable wellorders of the reals (or the Baire space $\omega_\omega$) and similarly, definable wellorders of $H(\kappa^+)$ are connected to definable wellorders of the generalized Baire space $\kappa_\kappa$.

**Theorem (Gödel, 1920ies)**

In $\mathbf{L}$, there is a (lightface) $\Sigma_1$-definable wellorder of $H(\kappa^+)$ for every infinite cardinal $\kappa$. 
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In $L$, there is a (lightface) $\Sigma_1$-definable wellorder of $H(\kappa^+)$ for every infinite cardinal $\kappa$.

Remark: Note that every $\Sigma_n$-definable wellordering $<$ is automatically $\Delta_n$-definable, because $x < y$ holds iff $x \neq y$ and $y \not< x$. 

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Observation (folklore?)

It is inconsistent with ZFC to have a ZF−-provably $\Delta_1$-definable wellorder of $H(\kappa^+)$ whenever $\kappa$ is an infinite cardinal.

Theorem (Mansfield, 1970)

The existence of a $\Sigma_1$-definable wellorder of $H(\omega_1)$ is equivalent to the statement that there is a real $x$ such that all reals are contained in $L[x]$.

Corollary

If there is a $\Sigma_1$-definable wellordering of $H(\omega_1)$, then CH holds.
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If there is a \( \Sigma_1 \)-definable wellordering of \( H(\omega_1) \), then CH holds.
Theorem (Friedman - Holy, 2011)

If $\kappa$ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$, then there is a cofinality-preserving forcing that introduces a $\Sigma_1$-definable wellordering of $H(\kappa^+)$ and preserves $2^\kappa = \kappa^+$. 
The GCH setting

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**Theorem (Asperó - Friedman, 2009)**

If \( \kappa \) is an uncountable cardinal with \( \kappa = \kappa^{<\kappa} \) and \( 2^\kappa = \kappa^+ \), then there is a cofinality-preserving forcing that introduces a lightface definable wellordering (of high complexity) of \( H(\kappa^+) \) and preserves \( 2^\kappa = \kappa^+ \).
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What if $2^{\kappa} > \kappa^+$?
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What if $2^{\kappa} > \kappa^+$?

Theorem (Asperó - Holy - Lücke, 2013)

The assumption $2^{\kappa} = \kappa^+$ can be dropped in the second theorem above, replacing preservation of $2^{\kappa} = \kappa^+$ by preservation of the value of $2^{\kappa}$. 
**Theorem (Friedman - Holy, 2011)**

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### Reminder (Mansfield)

If there is a $\Sigma_1$-definable wellordering of $H(\omega_1)$, then CH holds.
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Reminder (Mansfield)

If there is a $\Sigma_1$-definable wellordering of $H(\omega_1)$, then CH holds.

What about $\Sigma_1$-definable wellorderings of $H(\kappa^+)$ for uncountable $\kappa$?

Question

If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, does the existence of a $\Sigma_1$-definable wellordering of $H(\kappa^+)$ imply that $2^\kappa = \kappa^+$?
Almost Disjoint Coding

We will answer the above question negatively. To motivate our approach, we want to show how one can (quite easily) introduce $\Sigma_2$-definable wellorderings of $H(\kappa^+)$ when $\kappa$ is uncountable and $\kappa^\kappa = \kappa$.

Given some suitable enumeration $\langle s_\alpha \mid \alpha < \kappa \rangle$ of $\kappa^\kappa$, forcing with Solovay’s almost disjoint coding forcing makes a given set $A \subseteq \kappa^\kappa$ $\Sigma^0_2$-definable over $\kappa^\kappa$ - it adds a function $t: \kappa \to 2$ such that in the generic extension, for every $x \in \kappa^\kappa$,

\[ x \in A \iff \exists \beta < \kappa \ t(\alpha) = 1 \text{ for all } \beta < \alpha < \kappa \text{ with } s_\alpha \subseteq x. \]
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Given some suitable enumeration $\langle s_\alpha \mid \alpha < \kappa \rangle$ of $\kappa^\kappa$, forcing with Solovay’s almost disjoint coding forcing makes a given set $A \subseteq \kappa^\kappa \Sigma_2^0$-definable over $\kappa^\kappa$ - it adds a function $t : \kappa \rightarrow 2$ such that in the generic extension, for every $x \in \kappa^\kappa$,

$$x \in A \iff \exists \beta < \kappa \ t(\alpha) = 1 \text{ for all } \beta < \alpha < \kappa \text{ with } s_\alpha \subseteq x.$$ 

So we could pick any wellordering $<$ of $H(\kappa^+)$, code it by $A \subseteq \kappa^\kappa$ and make it $\Delta_1$-definable over $H(\kappa^+)$ of a $P$-generic extension. But forcing with $P$ adds new subsets of $\kappa$, so $<$ is not a wellordering of $H(\kappa^+)$ anymore.
Observation

If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, then there is a $<\kappa$-closed, $\kappa^+$-cc partial order $P \subseteq H(\kappa^+)$ that introduces a $\Sigma_2$-definable wellordering of $H(\kappa^+)$. 

Proof-Sketch: Pick any wellordering $\prec$ of $H(\kappa^+)$ and code it by $A \subseteq \kappa$. Apply the almost disjoint coding forcing (denote it by $P$) to make $A$ (and thus $\prec$) $\Delta_1$-definable over $H(\kappa^+)$. $P$ is $\kappa^+$-cc and $P \subseteq H(\kappa^+)$. This implies that every element $x$ of $H(\kappa^+)$ of the $P$-generic extension has a name $\dot{x}$ in the $H(\kappa^+)$ of the ground model. This allows us to define $x \prec \ast y \iff \exists \dot{x} \forall \dot{y} [\dot{x}_G = x \land \dot{y}_G = y \rightarrow \dot{x} \prec \dot{y}]$, where $G$ is the $P$-generic filter. Using $\Sigma_1$-definability of $P$ and $G$ over the new $H(\kappa^+)$, $\prec$ is a $\Sigma_2$-definable wellordering of the new $H(\kappa^+)$. □
Observation

If \( \kappa \) is an uncountable cardinal with \( \kappa^{<\kappa} = \kappa \), then there is a \( <\kappa \)-closed, \( \kappa^+ \)-cc partial order \( P \subseteq H(\kappa^+) \) that introduces a \( \Sigma_2 \)-definable wellordering of \( H(\kappa^+) \).

Proof-Sketch: Pick any wellordering \( < \) of \( H(\kappa^+) \) and code it by \( A \subseteq \kappa^\kappa \). Apply the almost disjoint coding forcing (denote it by \( P \)) to make \( A \) (and thus \( < \)) \( \Delta_1 \)-definable over \( H(\kappa^+) \). \( P \) is \( \kappa^+ \)-cc and \( P \subseteq H(\kappa^+) \).
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**Proof-Sketch:** Pick any wellordering $<$ of $H(\kappa^+)$ and code it by $A \subseteq {}^\kappa \kappa$. Apply the almost disjoint coding forcing (denote it by $P$) to make $A$ (and thus $<$) $\Delta_1$-definable over $H(\kappa^+)$. $P$ is $\kappa^+$-cc and $P \subseteq H(\kappa^+)$. This implies that every element $x$ of $H(\kappa^+)$ of the $P$-generic extension has a name $\dot{x}$ in the $H(\kappa^+)$ of the ground model.
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\[
x <^* y \iff \exists \dot{x} \forall \dot{y} \left( (\dot{x}^G = x \land \dot{y}^G = y) \rightarrow \dot{x} < \dot{y} \right),
\]

where \( G \) is the \( P \)-generic filter.
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If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, then there is a $<\kappa$-closed, $\kappa^+$-cc partial order $P \subseteq H(\kappa^+)$ that introduces a $\Sigma_2$-definable wellordering of $H(\kappa^+)$. 

Proof-Sketch: Pick any wellordering $<$ of $H(\kappa^+)$ and code it by $A \subseteq ^\kappa \kappa$. Apply the almost disjoint coding forcing (denote it by $P$) to make $A$ (and thus $<$) $\Delta_1$-definable over $H(\kappa^+)$. $P$ is $\kappa^+$-cc and $P \subseteq H(\kappa^+)$. This implies that every element $x$ of $H(\kappa^+)$ of the $P$-generic extension has a name $\dot{x}$ in the $H(\kappa^+)$ of the ground model. This allows us to define

$$x <^* y \iff \exists \dot{x} \forall \dot{y} \left[ (\dot{x}^G = x \land \dot{y}^G = y) \rightarrow \dot{x} < \dot{y} \right],$$

where $G$ is the $P$-generic filter. Using $\Sigma_1$-definability of $P$ and $G$ over the new $H(\kappa^+)$, $<^*$ is a $\Sigma_2$-definable wellordering of the new $H(\kappa^+)$.

$\square$
If $2^\kappa = \kappa^+$, it is possible to pull a small trick and spare one quantifier in the above (by coding all initial segments of $<$, which in that case have size at most $\kappa$ and are thus elements of $H(\kappa^+)$). Otherwise however, the above suggests that one cannot hope for a wellordering of the $H(\kappa^+)$ of the ground model to *induce* a $\Sigma_1$-definable wellordering of the $H(\kappa^+)$ of some generic extension, at least not *directly* via names.
By different means, we obtained the following.

**Theorem (Holy - Lücke, 2014)**

If $\kappa$ is an uncountable cardinal with $\kappa^{\lessdot \kappa} = \kappa$ and $2^\kappa$ regular then there is a partial order $P$ which is $\lessdot \kappa$-closed and preserves cofinalities $\leq 2^\kappa$ and the value of $2^\kappa$ and introduces a $\Sigma_1$-definable wellordering of $H(\kappa^+)$. Moreover, $P$ introduces a $\Delta_1$ Bernstein subset of $\kappa \kappa$, i.e. a subset $X$ of $\kappa \kappa$ such that neither $X$ nor its complement contain a perfect subset of $\kappa \kappa$. 

The basic idea of our solution is to build a forcing $P$ that adds a wellordering of $H(\kappa^+)$ of the $P$-generic extension (using initial segments (represented in the ground model as sequences of $P$-names) as conditions) and simultaneously makes this wellordering definable.
By different means, we obtained the following.

**Theorem (Holy - Lücke, 2014)**

If \( \kappa \) is an uncountable cardinal with \( \kappa^\kappa = \kappa \) and \( 2^\kappa \) regular then there is a partial order \( P \) which is \( \kappa \)-closed and preserves cofinalities \( \leq 2^\kappa \) and the value of \( 2^\kappa \) and introduces a \( \Sigma_1 \)-definable wellordering of \( H(\kappa^+) \).

Moreover, \( P \) introduces a \( \Delta_1 \) Bernstein subset of \( \kappa^\kappa \), i.e. a subset \( X \) of \( \kappa^\kappa \) such that neither \( X \) nor its complement contain a perfect subset of \( \kappa^\kappa \).

The basic idea of our solution is to build a forcing \( P \) that adds a wellordering of \( H(\kappa^+) \) of the \( P \)-generic extension (using initial segments (represented in the ground model as sequences of \( P \)-names) as conditions) and simultaneously makes this wellordering definable.
Let $\lambda = 2^\kappa$. We inductively construct a sequence $\langle P_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders with the property that $P_\delta$ is a complete subforcing of $P_\gamma$ whenever $\delta \leq \gamma \leq \lambda$ (i.e. an iteration of length $\lambda$) and let $P = P_\lambda$. 
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Let $\lambda = 2^\kappa$. We inductively construct a sequence $\langle P_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders with the property that $P_\delta$ is a complete subforcing of $P_\gamma$ whenever $\delta \leq \gamma \leq \lambda$ (i.e. an iteration of length $\lambda$) and let $P = P_\lambda$. Assume we have constructed $P_\delta$ for every $\delta < \gamma$.

A condition $p$ in $P_\gamma$ specifies a sequence $\vec{A}_p$ of length at most $\gamma$ where for every $\delta < \gamma$, $\vec{A}_p(\delta)$ is a nice $P_\delta$-name for a subset of $\kappa$ and whenever $\bar{\gamma} < \gamma$, $p \upharpoonright \bar{\gamma}$ forces that $\langle \vec{A}_p(\delta) \mid \delta \leq \bar{\gamma} \rangle$ is a sequence of codes for pairwise distinct elements of $H(\kappa^+)$. 
Let $\lambda = 2^{\kappa}$. We inductively construct a sequence $\langle P_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders with the property that $P_\delta$ is a complete subforcing of $P_\gamma$ whenever $\delta \leq \gamma \leq \lambda$ (i.e. an iteration of length $\lambda$) and let $P = P_\lambda$. Assume we have constructed $P_\delta$ for every $\delta < \gamma$.

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$p$ also specifies coding components $\vec{c}_p$ of size $< \kappa$ such that $\vec{c}_p$ is a condition in $C(A_p)$ where $A_p$ is $\vec{A}_p$ "restricted" to $a_p$ (which we require to be decided by $p$ hence $A_p \in V$).
Let $\lambda = 2^\kappa$. We inductively construct a sequence $\langle P_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders with the property that $P_\delta$ is a complete subforcing of $P_\gamma$ whenever $\delta \leq \gamma \leq \lambda$ (i.e. an iteration of length $\lambda$) and let $P = P_\lambda$. Assume we have constructed $P_\delta$ for every $\delta < \gamma$.

A condition $p$ in $P_\gamma$ specifies a sequence $\vec{A}_p$ of length at most $\gamma$ where for every $\delta < \gamma$, $\vec{A}_p(\delta)$ is a nice $P_\delta$-name for a subset of $\kappa$ and whenever $\bar{\gamma} < \gamma$, $p \upharpoonright \bar{\gamma}$ forces that $\langle \vec{A}_p(\delta) \mid \delta \leq \bar{\gamma} \rangle$ is a sequence of codes for pairwise distinct elements of $H(\kappa^+)$. $p$ also specifies $a_p$, a subset of $\lambda \times \kappa$ of size less than $\kappa$ and for $p$ to be a condition in $P_\gamma$ we in fact require that whenever $(\delta, \alpha) \in a_p$ then $p \upharpoonright \delta$ decides whether $\alpha \in \vec{A}_p(\delta)$.

Moreover we will define a coding forcing $C(A)$ that is capable of coding a subset $A$ of $\lambda$ by a generically added subset of $\kappa$ in a $\Sigma_1$-way over $H(\kappa^+)$ with the property that if $B \supseteq A$ then $C(A)$ is a complete subforcing of $C(B)$. The above $p$ also specifies coding components $\vec{c}_p$ of size $< \kappa$ such that $\vec{c}_p$ is a condition in $C(A_p)$ where $A_p$ is $\vec{A}_p$ “restricted” to $a_p$ (which we require to be decided by $p$ hence $A_p \in V$).
Remember: $p \in P_\gamma$ for $\gamma \leq \lambda$ is of the form $p = (\vec{A}_p, a_p, \vec{c}_p)$. $q \in P_\gamma$ is stronger than $p$ if $\vec{A}_q$ end-extends $\vec{A}_p$, $a_q$ is a superset of $a_p$ and $\vec{c}_q$ extends $\vec{c}_p$ in the forcing $C(A_q)$.

Let $G$ be $P_\lambda$-generic, let $\vec{A} = \bigcup_{p \in G} \vec{A}_p$. Density arguments show that $\vec{A}^G$ is a $\lambda$-sequence of codes for elements of $H(\kappa^+)$ of $V[G]$ that gives rise to an injective enumeration of $H(\kappa^+)$ of $V[G]$, for it can be shown that every element of $H(\kappa^+)$ of $V[G]$ is added by $P_\gamma$ for some $\gamma < \lambda$. 
Remember: $p \in P_{\gamma}$ for $\gamma \leq \lambda$ is of the form $p = (\vec{A}_p, a_p, \vec{c}_p)$. $q \in P_{\gamma}$ is stronger than $p$ if $\vec{A}_q$ end-extends $\vec{A}_p$, $a_q$ is a superset of $a_p$ and $\vec{c}_q$ extends $\vec{c}_p$ in the forcing $C(A_q)$.

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Remember: $p \in P_\gamma$ for $\gamma \leq \lambda$ is of the form $p = (\bar{A}_p, a_p, \bar{c}_p)$. $q \in P_\gamma$ is stronger than $p$ if $\bar{A}_q$ end-extends $\bar{A}_p$, $a_q$ is a superset of $a_p$ and $\bar{c}_q$ extends $\bar{c}_p$ in the forcing $C(A_q)$.

Let $G$ be $P_\lambda$-generic, let $\bar{A} = \bigcup_{p \in G} \bar{A}_p$. Density arguments show that $\bar{A}^G$ is a $\lambda$-sequence of codes for elements of $H(\kappa^+)$ of $V[G]$ that gives rise to an injective enumeration of $H(\kappa^+)$ of $V[G]$, for it can be shown that every element of $H(\kappa^+)$ of $V[G]$ is added by $P_\gamma$ for some $\gamma < \lambda$. Moreover $\bigcup_{p \in G} a_p = \lambda \times \kappa$, i.e. there is a generic subset of $\kappa$ that codes $\bar{A}^G$ and we obtain that $\bar{A}^G$ is $\Sigma_1$-definable over $H(\kappa^+)^{V[G]}$.

Since $\bar{A}^G$ is an enumeration of $H(\kappa^+)^{V[G]}$ in order-type $\lambda$, we have produced a $\Sigma_1$-definable wellordering of $H(\kappa^+)^{V[G]}$. Of course the above doesn’t quite make sense, as we have not yet specified the coding forcing $C(A_q)$. 

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Simplest Possible Wellorders
January 26, 2014 11 / 17
Remember: \( p \in P_\gamma \) for \( \gamma \leq \lambda \) is of the form \( p = (\vec{A}_p, a_p, \vec{c}_p) \). \( q \in P_\gamma \) is stronger than \( p \) if \( \vec{A}_q \) end-extends \( \vec{A}_p \), \( a_q \) is a superset of \( a_p \) and \( \vec{c}_q \) extends \( \vec{c}_p \) in the forcing \( C(A_q) \).

Let \( G \) be \( P_\lambda \)-generic, let \( \vec{A} = \bigcup_{p \in G} \vec{A}_p \). Density arguments show that \( \vec{A}^G \) is a \( \lambda \)-sequence of codes for elements of \( H(\kappa^+) \) of \( V[G] \) that gives rise to an injective enumeration of \( H(\kappa^+) \) of \( V[G] \), for it can be shown that every element of \( H(\kappa^+) \) of \( V[G] \) is added by \( P_\gamma \) for some \( \gamma < \lambda \). Moreover \( \bigcup_{p \in G} a_p = \lambda \times \kappa \), i.e. there is a generic subset of \( \kappa \) that codes \( \vec{A}^G \) and we obtain that \( \vec{A}^G \) is \( \Sigma_1 \)-definable over \( H(\kappa^+)^{V[G]} \).

Since \( \vec{A}^G \) is an enumeration of \( H(\kappa^+)^{V[G]} \) in order-type \( \lambda \), we have produced a \( \Sigma_1 \)-definable wellordering of \( H(\kappa^+)^{V[G]} \).

Of course the above doesn’t quite make sense, as we have not yet specified the coding forcing \( C(A) \).
Club Coding

joint work with David Asperó and Philipp Lücke
The Coding Forcing

We need a forcing that codes a given \( A \subseteq \lambda = 2^{\kappa} \) by a generically added subset of \( \kappa \). This could be achieved using the Almost Disjoint Coding forcing. However to obtain the desired property that \( P_{\gamma_0} \) is a complete subforcing of \( P_{\gamma_1} \) whenever \( \gamma_0 < \gamma_1 \), we need our coding forcing \( C \) to have the following property:

\[ (*) \text{ If } A \subseteq B \subseteq \lambda, \text{ } C(A) \text{ is a complete subforcing of } C(B). \]
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\begin{equation}
(*) \text{ If } A \subseteq B \subseteq \lambda, \ C(A) \text{ is a complete subforcing of } C(B). \end{equation}

This requirement is not satisfied by the Almost Disjoint Coding forcing.
We need a forcing that codes a given $A \subseteq \lambda = 2^\kappa$ by a generically added subset of $\kappa$. This could be achieved using the Almost Disjoint Coding forcing. However to obtain the desired property that $P_{\gamma_0}$ is a complete subforcing of $P_{\gamma_1}$ whenever $\gamma_0 < \gamma_1$, we need our coding forcing $C$ to have the following property:

\[(*)\text{ If } A \subseteq B \subseteq \lambda, \text{ then } C(A) \text{ is a complete subforcing of } C(B).\]

This requirement is not satisfied by the Almost Disjoint Coding forcing. We thus choose $C(A)$ to be a variation of the Almost Disjoint Coding forcing for $A$ (that could in fact rather be seen as a variation of the Canonical Function Coding by Asperó and Friedman), that combines the classic forcing with iterated club shooting and has the desired property that $A \subseteq B$ implies that $C(A)$ is a complete subforcing of $C(B)$.
Definition (Asperó-Holy-Lücke, 2013)

Given $A \subseteq \kappa$, we let $C(A)$ be the partial order whose conditions are tuples

$$p = (s_p, t_p, \langle c^p_x | x \in a_p \rangle)$$

such that the following hold for some successor ordinal $\gamma_p < \kappa$.

1. $s_p : \gamma_p \to ^{<\kappa}\kappa$, $t_p : \gamma_p \to 2$ and $a_p \in [A]^{<\kappa}$.
2. If $x \in a_p$, then $c^p_x$ is a closed subset of $\gamma_p$ and

$$s_p(\alpha) \subseteq x \to t_p(\alpha) = 1$$

for all $\alpha \in c^p_x$.

We let $q \leq p$ if $s_p = s_q \upharpoonright \gamma_p$, $t_p = t_q \upharpoonright \gamma_p$, $a_p \subseteq a_q$ and $c^p_x = c^q_x \cap \gamma_p$ for every $x \in a_p$. 

Lemma (Asperó-Holy-Lücke, 2013)

Assume $G$ is $C(A)$-generic, $s = \bigcup_{p \in G} s_p$ and $t = \bigcup_{p \in G} t_p$. Then $s : \kappa \to ^\kappa \kappa$, $t : \kappa \to 2$ and $A$ is equal to the set of all $x \in (\kappa \kappa)^{V[G]}$ such that

$$\forall \alpha \in C \ [s(\alpha) \subseteq x \to t(\alpha) = 1]$$

holds for some club subset $C$ of $\kappa$ in $V[G]$.

Moreover, $C(A)$ is $<\kappa$-closed, $\kappa^+$-cc, a subset of $H(\kappa^+)$ and whenever $A \subseteq B \subseteq ^\kappa \kappa$, then $C(A)$ is a complete subforcing of $C(B)$. 
Theorem (Holy - Lücke, 2014)

If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and $2^\kappa$ regular then there is a partial order $P$ which is $<\kappa$-closed and preserves cofinalities $\leq 2^\kappa$ and the value of $2^\kappa$ and introduces a $\Sigma_1$-definable wellordering of $H(\kappa^+)$. 

Corollary (Holy - Lücke, 2014)

If $\kappa$ is a regular uncountable $L$-cardinal, then there is a cofinality-preserving forcing extension of $L$ with a $\Sigma_1(\kappa)$-definable wellorder of $H(\kappa^+)$ and $2^\kappa > \kappa^+$. 

Peter Holy (Bristol) 
Simplest Possible Wellorders 
January 26, 2014 16 / 17
Theorem (Holy - Lücke, 2014)

If \( \kappa \) is an uncountable cardinal with \( \kappa^{<\kappa} = \kappa \) and \( 2^\kappa \) regular then there is a partial order \( P \) which is \( <\kappa \)-closed and preserves cofinalities \( \leq 2^\kappa \) and the value of \( 2^\kappa \) and introduces a \( \Sigma_1 \)-definable wellordering of \( H(\kappa^+) \).

If \( \kappa = \lambda^+ \) and \( \lambda^{<\lambda} = \lambda \), one can improve the above to a \( \Sigma_1 \)-definable wo that only uses a parameter from the ground model, basically by coding, during the above construction, the parameter into the stationarity pattern of a ground model \( \kappa \)-seq. of disjoint stationary subsets of \( \kappa \) on \( \text{cof}(\lambda) \).
Theorem (Holy - Lücke, 2014)

If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and $2^\kappa$ regular then there is a partial order $P$ which is $<\kappa$-closed and preserves cofinalities $\leq 2^\kappa$ and the value of $2^\kappa$ and introduces a $\Sigma_1$-definable wellordering of $H(\kappa^+)$. If $\kappa = \lambda^+$ and $\lambda^{<\lambda} = \lambda$, one can improve the above to a $\Sigma_1$-definable wo that only uses a parameter from the ground model, basically by coding, during the above construction, the parameter into the stationarity pattern of a ground model $\kappa$-seq. of disjoint stationary subsets of $\kappa$ on $\text{cof}(\lambda)$. If sufficiently close to $L$, one may choose a canonically $\Sigma_1(\kappa)$-definable such sequence of stationary subsets of $\kappa$ and obtain a $\Sigma_1(\kappa)$-definable wellorder of $H(\kappa^+)$. Similar results are possible for inaccessible $\kappa$, but one needs to assume the existence of a $\kappa$-sequence of disjoint fat stationary subsets of $\kappa$. 

Corollary (Holy - Lücke, 2014)

If $\kappa$ is a regular uncountable $L$-cardinal, then there is a cofinality-preserving forcing extension of $L$ with a $\Sigma_1(\kappa)$-definable wellorder of $H(\kappa^+)$ and $2^\kappa > \kappa^+$. 

Peter Holy (Bristol)
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Theorem (Holy - Lücke, 2014)

If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and $2^\kappa$ regular then there is a partial order $P$ which is $<\kappa$-closed and preserves cofinalities $\leq 2^\kappa$ and the value of $2^\kappa$ and introduces a $\Sigma_1$-definable wellordering of $H(\kappa^+)$. 

If $\kappa = \lambda^+$ and $\lambda^{<\lambda} = \lambda$, one can improve the above to a $\Sigma_1$-definable wo that only uses a parameter from the ground model, basically by coding, during the above construction, the parameter into the stationarity pattern of a ground model $\kappa$-seq. of disjoint stationary subsets of $\kappa$ on $\text{cof}(\lambda)$. If sufficiently close to $L$, one may choose a canonically $\Sigma_1(\kappa)$-definable such sequence of stationary subsets of $\kappa$ and obtain a $\Sigma_1(\kappa)$-definable wellorder of $H(\kappa^+)$. Similar results are possible for inaccessible $\kappa$, but one needs to assume the existence of a $\kappa$-sequence of disjoint fat stationary subsets of $\kappa$.

Corollary (Holy - Lücke, 2014)

If $\kappa$ is a regular uncountable $L$-cardinal, then there is a cofinality-preserving forcing extension of $L$ with a $\Sigma_1(\kappa)$-definable wellorder of $H(\kappa^+)$ and $2^\kappa > \kappa^+$. 
Thank you.