Simplest Possible Wellorders of $H(\kappa^+)$

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presenting joint work with Philipp Lücke

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Basic Motivation

Question

How simple a wellordering of $H(\kappa^+)$ can one have definably (by a first order formula in the language of set theory) over $H(\kappa^+)$?

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We want to measure complexity in terms of the standard Lévy hierarchy and in terms of the necessary parameters. Note that definable wellorders of $H(\omega_1)$ are closely connected to definable wellorders of the reals (or the Baire space ${}^\omega\omega$) and similarly, definable wellorders of $H(\kappa^+)$ are connected to definable wellorders of the generalized Baire space ${}^\kappa\kappa$.

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In $\mathbf{L}$*, there is a (lightface)* Σ_1*-definable wellorder of* $H(\kappa^+)$ *for every infinite cardinal* κ*.*

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Theorem (Gödel, 1920ies)

In $\mathbf{L}$, there is a (lightface) Σ_1-definable wellorder of $H(\kappa^+)$ for every infinite cardinal κ.

Remark: Note that every $\mathbf{\Sigma}_n$-definable wellordering $<$ is automatically $\mathbf{\Delta}_n$-definable, because $x < y$ holds iff $x \neq y$ and $y \not< x$.

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Theorem (Mansfield, 1970)

The existence of a $\mathbf{\Sigma}_1$*-definable wellorder of* $H(\omega_1)$ *is equivalent to the statement that there is a real* x *such that all reals are contained in* $\mathbf{L}[x]$*.*

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Corollary

If there is a $\boldsymbol{\Sigma}_1$*-definable wellordering of* $H(\omega_1)$*, then* CH *holds.*

The GCH setting

Theorem (Friedman - Holy, 2011)

If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$, then there is a cofinality-preserving forcing that introduces a $\mathbf{\Sigma}_1$-definable wellordering of $H(\kappa^+)$ and preserves $2^\kappa = \kappa^+$.

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If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$, then there is a cofinality-preserving forcing that introduces a lightface definable wellordering (of high complexity) of $H(\kappa^+)$ and preserves $2^\kappa = \kappa^+$.

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What if $2^\kappa > \kappa^+$?

Theorem (Asperó - Holy - Lücke, 2013)

The assumption $2^\kappa = \kappa^+$ can be dropped in the second theorem above, replacing preservation of $2^\kappa = \kappa^+$ by preservation of the value of 2^κ.

$\mathbf{\Sigma}_1$ and non-GCH

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Reminder (Mansfield)

If there is a $\mathbf{\Sigma}_1$-definable wellordering of $H(\omega_1)$, then CH holds.

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If there is a $\boldsymbol{\Sigma}_1$-definable wellordering of $H(\omega_1)$, then CH holds.

What about $\boldsymbol{\Sigma}_1$-definable wellorderings of $H(\kappa^+)$ for uncountable κ?

Question

If κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, does the existence of a $\boldsymbol{\Sigma}_1$-definable wellordering of $H(\kappa^+)$ imply that $2^\kappa = \kappa^+$?

Almost Disjoint Coding

We will answer the above question negatively. To motivate our approach, we want to show how one can (quite easily) introduce $\mathbf{\Sigma}_2$-definable wellorderings of $H(\kappa^+)$ when κ is uncountable and $\kappa^{<\kappa} = \kappa$.

Given some suitable enumeration $\langle s_\alpha \mid \alpha < \kappa \rangle$ of ${}^{<\kappa}\kappa$, forcing with Solovay's almost disjoint coding forcing makes a given set $A \subseteq {}^\kappa\kappa$ $\mathbf{\Sigma}^0_2$-definable over ${}^\kappa\kappa$ - it adds a function $t \colon \kappa \to 2$ such that in the generic extension, for every $x \in {}^\kappa\kappa$,

$$x \in A \iff \exists \beta < \kappa \ t(\alpha) = 1 \text{ for all } \beta < \alpha < \kappa \text{ with } s_\alpha \subseteq x.$$

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$$x \in A \iff \exists \beta < \kappa \; t(\alpha) = 1 \text{ for all } \beta < \alpha < \kappa \text{ with } s_\alpha \subseteq x.$$

So we could pick any wellordering $<$ of $H(\kappa^+)$, code it by $A \subseteq {}^{\kappa}\kappa$ and make it $\mathbf{\Delta}_1$-definable over $H(\kappa^+)$ of a P-generic extension. But forcing with P adds new subsets of κ, so $<$ is not a wellordering of $H(\kappa^+)$ anymore.

$\mathbf{\Sigma}_2$-definable Wellorderings

Observation

If κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, then there is a $<\kappa$-closed, κ^+-cc partial order $P \subseteq H(\kappa^+)$ that introduces a $\mathbf{\Sigma}_2$-definable wellordering of $H(\kappa^+)$.

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Proof-Sketch: Pick any wellordering $<$ of $H(\kappa^+)$ and code it by $A \subseteq {}^\kappa\kappa$. Apply the almost disjoint coding forcing (denote it by P) to make A (and thus $<$) $\mathbf{\Delta}_1$-definable over $H(\kappa^+)$. P is κ^+-cc and $P \subseteq H(\kappa^+)$.

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$$x <^* y \iff \exists \dot{x} \forall \dot{y} \left[(\dot{x}^G = x \wedge \dot{y}^G = y) \rightarrow \dot{x} < \dot{y} \right],$$

where G is the P-generic filter.

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$$x <^* y \iff \exists \dot{x} \forall \dot{y} \left[(\dot{x}^G = x \wedge \dot{y}^G = y) \rightarrow \dot{x} < \dot{y} \right],$$

where G is the P-generic filter. Using $\mathbf{\Sigma}_1$-definability of P and G over the new $H(\kappa^+)$, $<^*$ is a $\mathbf{\Sigma}_2$-definable wellordering of the new $H(\kappa^+)$. $\square$

$\mathbf{\Sigma}_1$?

If $2^\kappa = \kappa^+$, it is possible to pull a small trick and spare one quantifier in the above (by coding all initial segments of $<$, which in that case have size at most κ and are thus elements of $H(\kappa^+)$). Otherwise however, the above suggests that one cannot hope for a wellordering of the $H(\kappa^+)$ of the ground model to *induce* a $\mathbf{\Sigma}_1$-definable wellordering of the $H(\kappa^+)$ of some generic extension, at least not *directly* via names.

Our Theorem

By different means, we obtained the following.

Theorem (Holy - Lücke, 2014)

If κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and 2^κ regular then there is a partial order P which is $<\kappa$-closed and preserves cofinalities $\leq 2^\kappa$ and the value of 2^κ and introduces a $\mathbf{\Sigma}_1$-definable wellordering of $H(\kappa^+)$.

Moreover, P introduces a $\mathbf{\Delta}_1$ Bernstein subset of ${}^\kappa\kappa$, i.e. a subset X of ${}^\kappa\kappa$ such that neither X nor its complement contain a perfect subset of ${}^\kappa\kappa$.

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The basic idea of our solution is to build a forcing P that adds a wellordering of $H(\kappa^+)$ of the P-generic extension (using initial segments (represented in the ground model as sequences of P-names) as conditions) and simultaneously makes this wellordering definable.

Let $\lambda = 2^\kappa$. We inductively construct a sequence $\langle P_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders with the property that P_δ is a complete subforcing of P_γ whenever $\delta \leq \gamma \leq \lambda$ (i.e. an iteration of length λ) and let $P = P_\lambda$.

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A condition p in P_γ specifies a sequence $\vec{A}_p$ of length at most γ where for every $\delta < \gamma$, $\vec{A}_p(\delta)$ is a nice P_δ-name for a subset of κ and whenever $\bar{\gamma} < \gamma$, $p \restriction \bar{\gamma}$ forces that $\langle \vec{A}_p(\delta) \mid \delta \leq \bar{\gamma} \rangle$ is a sequence of codes for pairwise distinct elements of $H(\kappa^+)$.

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Moreover we will define a coding forcing $C(A)$ that is capable of coding a subset A of λ by a generically added subset of κ in a Σ_1-way over $H(\kappa^+)$ with the property that if $B \supseteq A$ then $C(A)$ is a complete subforcing of $C(B)$. The above p also specifies coding components $\vec{c}_p$ of size $< \kappa$ such that $\vec{c}_p$ is a condition in $C(A_p)$ where A_p is $\vec{A}_p$ "restricted" to a_p (which we require to be decided by p hence $A_p \in V$).

Remember: $p \in P_\gamma$ for $\gamma \leq \lambda$ is of the form $p = (\vec{A}_p, a_p, \vec{c}_p)$. $q \in P_\gamma$ is stronger than p if $\vec{A}_q$ end-extends $\vec{A}_p$, a_q is a superset of a_p and $\vec{c}_q$ extends $\vec{c}_p$ in the forcing $C(A_q)$.

Let G be P_λ-generic, let $\vec{A} = \bigcup_{p \in G} \vec{A}_p$. Density arguments show that $\vec{A}^G$ is a λ-sequence of codes for elements of $H(\kappa^+)$ of $V[G]$ that gives rise to an injective enumeration of $H(\kappa^+)$ of $V[G]$, for it can be shown that every element of $H(\kappa^+)$ of $V[G]$ is added by P_γ for some $\gamma < \lambda$.

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Since $\vec{A}^G$ is an enumeration of $H(\kappa^+)^{V[G]}$ in order-type λ, we have produced a $\mathbf{\Sigma}_1$-definable wellordering of $H(\kappa^+)^{V[G]}$.

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Of course the above doesn't quite make sense, as we have not yet specified the coding forcing $C(A)$.

Club Coding

joint work with David Asperó and Philipp Lücke

The Coding Forcing

We need a forcing that codes a given $A \subseteq \lambda = 2^\kappa$ by a generically added subset of κ. This could be achieved using the Almost Disjoint Coding forcing. However to obtain the desired property that P_{γ_0} is a complete subforcing of P_{γ_1} whenever $\gamma_0 < \gamma_1$, we need our coding forcing C to have the following property:

(*) If $A \subseteq B \subseteq \lambda$, $C(A)$ is a complete subforcing of $C(B)$.

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We thus choose $C(A)$ to be a variation of the Almost Disjoint Coding forcing for A (that could in fact rather be seen as a variation of the Canonical Function Coding by Asperó and Friedman), that combines the classic forcing with iterated club shooting and has the desired property that $A \subseteq B$ implies that $C(A)$ is a complete subforcing of $C(B)$.

Club Coding

Definition (Asperó-Holy-Lücke, 2013)

Given $A \subseteq {}^{\kappa}\kappa$, we let $C(A)$ be the partial order whose conditions are tuples

$$p = (s_p, t_p, \langle c_x^p \mid x \in a_p \rangle)$$

such that the following hold for some successor ordinal $\gamma_p < \kappa$.

1. $s_p \colon \gamma_p \to {}^{<\kappa}\kappa$, $t_p \colon \gamma_p \to 2$ and $a_p \in [A]^{<\kappa}$.
2. If $x \in a_p$, then c_x^p is a closed subset of γ_p and

$$s_p(\alpha) \subseteq x \to t_p(\alpha) = 1$$

   for all $\alpha \in c_x^p$.

We let $q \leq p$ if $s_p = s_q \restriction \gamma_p$, $t_p = t_q \restriction \gamma_p$, $a_p \subseteq a_q$ and $c_x^p = c_x^q \cap \gamma_p$ for every $x \in a_p$.

Club Coding

Lemma (Asperó-Holy-Lücke, 2013)

Assume G *is* $C(A)$*-generic,* $s = \bigcup_{p \in G} s_p$ *and* $t = \bigcup_{p \in G} t_p$*. Then* $s \colon \kappa \longrightarrow {}^{<\kappa}\kappa$*,* $t \colon \kappa \longrightarrow 2$ *and* A *is equal to the set of all* $x \in ({}^{\kappa}\kappa)^{V[G]}$ *such that*

$$\forall \alpha \in C \ \ [s(\alpha) \subseteq x \rightarrow t(\alpha) = 1]$$

holds for some club subset C *of* κ *in* $V[G]$*.*

Moreover, $C(A)$ *is* $<\kappa$*-closed,* κ^+*-cc, a subset of* $H(\kappa^+)$ *and whenever* $A \subseteq B \subseteq {}^{\kappa}\kappa$*, then* $C(A)$ *is a complete subforcing of* $C(B)$*.*

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If $\kappa = \lambda^+$ and $\lambda^{<\lambda} = \lambda$, one can improve the above to a Σ_1-definable wo that only uses a parameter from the ground model, basically by coding, during the above construction, the parameter into the stationarity pattern of a ground model κ-seq. of disjoint stationary subsets of κ on $\mathrm{cof}(\lambda)$.

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Corollary (Holy - Lücke, 2014)

If κ is a regular uncountable $\mathbf{L}$-cardinal, then there is a cofinality-preserving forcing extension of $\mathbf{L}$ with a $\Sigma_1(\kappa)$-definable wellorder of $H(\kappa^+)$ and $2^\kappa > \kappa^+$.

Thank you.