Peter Holy - Simplest Possible Wellorders of $H(\kappa^+)$

$H(\kappa^+)$ is the collection of all sets of hereditary cardinality at most κ. We investigate how simple a wellordering of $H(\kappa^+)$ one can have definably (by a first order formula in the language of set theory) over $H(\kappa^+)$. Thereby, we will measure complexity of the defining formulas in terms of the standard Lévy hierarchy and in terms of the necessary parameters.

By Gödel's classic result on the consistency of AC, $H(\kappa^+)$ has a Σ_1-definable wellorder for every infinite cardinal κ in $\mathbf{L}$. Can we get something similar *outside of* $\mathbf{L}$, for example if 2^κ (and thus $H(\kappa^+)$) is large?

For $H(\omega_1)$, we have the following.

Theorem 1 (Mansfield, 1970) *The existence of a* $\boldsymbol{\Sigma}_1$*-definable wellorder of* $H(\omega_1)$ *is equivalent to the statement that there is a real* x *such that all reals are contained in* $\mathbf{L}[x]$.

We show that unlike for $H(\omega_1)$, if κ is uncountable with $\kappa^{<\kappa} = \kappa$, one can obtain $\boldsymbol{\Sigma}_1$-definable wellorders of $H(\kappa^+)$ in *nice* generic extensions of the universe (i.e. in models *far from* $\mathbf{L}$, for example in models where 2^κ is large).

Under some additional assumptions on the ground model, one can also assure that the parameters used in the defining formulas are *simple*: Under mild additional assumptions, one can use a parameter from the ground model only; Under strong additional assumptions, one can make sure that only κ is needed as parameter (those strong assumptions still allow for large 2^κ).

This is joint work with Philipp Lücke.