An example of a rigid superuniversal metric space

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Definition

We say that a metric space X is κ-*homogeneous*, if for every $A, B \in [X]^{<\kappa}$ and every isometry $f_0 : A \to B$ there is an isometry $f : X \to X$ such that $f \restriction A = f_0$.
We say that a metric space X is [strongly] κ-*universal* if every metric space Y such that $|Y| \leqslant \kappa$ $[w(Y) \leqslant \kappa]$ can be isometrically embedded into X.

Theorem (Katětov, 1986)

For every uncountable κ such that $\kappa = \sup\{\kappa^\lambda : \lambda < \kappa\}$ there exists a unique (up to isometry) κ-homogeneous and strongly κ-universal metric space.

Introduction

Definition (Hechler)

Let κ be an uncountable cardinal. We say that a metric space X is *κ-superuniversal* if for every metric space Y of cardinality at most κ and every isometric embedding $f_0 : Y_0 \to X$, where $Y_0 \in [Y]^{<\kappa}$, there is an isometric embedding $f : Y \to X$ such that $f \restriction Y_0 = f_0$.

Theorem (Hechler, 1973)

For every uncountable regular cardinal κ there exists a κ-superuniversal metric space of cardinality $\sum_{\lambda<\kappa} 2^{\lambda}$.

Remark

Every κ-superuniversal metric space of cardinality κ is κ-homogeneous.

Main result

Kubiś suggested that there should exists a κ-superuniversal metric space which is not κ-homogeneous.
It occurs that κ-superuniversal space can be rigid (i.e. has only one isometry, its identity function):

Theorem (W.B.)

Assume that κ is a regular cardinal such that $\lambda^\omega < \kappa$ for every $\lambda < \kappa$. Then there exists a rigid κ-superuniversal metric space.

Adding points to a space

κ-superuniversality can be achieved by adding to a metric space X a family of points $\{x_\alpha : \alpha < \lambda\}$ and defining a metric d on

$$K(X) = X \cup \{x_\alpha : \alpha < \lambda\}$$

such that if $f_0 : Y \setminus \{y\} \to X$ is an isometric embedding, where $|Y| < \kappa$, then there is x_α such that there is an isometry $f : Y \to X \cup \{x_\alpha\}$ such that $f \restriction (Y \setminus \{y\}) = f_0$.

These points can be considered as *Katětov's maps*. Defining

$$K^{\beta+1}(X) = K(K^\beta(X)) \text{ and } K^\alpha(X) = \bigcup_{\beta<\alpha} K^\beta(X) \text{ for } \alpha \text{ limit,}$$

we see that $K^\kappa(X)$ is κ-superuniversal.

Unitary spaces and weak middle points

Definition (Hechler)

We say that a metric space (X, d) is *unitary* if $d(x, y) = 1$ for every $x, y \in X$, $x \neq y$.

A unitary subspace $D \subseteq X$ can have a *weak middle point*, i.e. a point $x \in X$ such that $d(x, y) = d(x, y') < 1$ for all $y, y' \in D$. The idea of obtaining the rigidity was to define *unitary character of* $x \in X$ by

$$\tau_w(x, X) = \sup\{|D| : D \in \mathcal{D}_w(x, X)\},$$

where

$$\mathcal{D}_w(x, X) = \{D \subseteq X : D \text{ is unitary, } x \in D \text{ and there is no weak middle point of any } D' \in [D]^\kappa\}.$$

Adding sufficiently many unitary subspaces, we would be able to show that $\tau_w(x, X) \neq \tau_w(y, X)$ for $x \neq y$.

Adding unitary subspaces

Assume that X is a subspace of a metric space Y, and each $x \in X$ has its own unitary subspace $D_x \subseteq X$. Then for the set $\{y_\alpha : \alpha < \lambda\} = Y \setminus X$ we consider a family $\{D_\alpha : \alpha < \lambda\}$ of unitary spaces such that

- $|D_\alpha| > |X|$,
- if $\alpha \neq \beta$ then $D_\alpha \cap D_\beta = \emptyset$,
- $D_\alpha \cap Y = \{y_\alpha\}$.

There exists a metric d on the set $A(Y, X) = Y \cup \bigcup \{D_\alpha : \alpha < \lambda\}$ such that D_α and Y are subspaces of $A(Y, X)$, and

$$d(w, v) = d(w, y_\alpha) + d(y_\alpha, y_\beta) + d(y_\beta, v)$$

for all $w \in D_\alpha$ and $v \in D_\beta$. We see that $D_\alpha \in \mathcal{D}_w(y_\alpha, A(Y, X))$.

Merging unitary subspaces and Katětov maps

We start with a metric space X_0 such that $|X_0| \leqslant \kappa$. We define

$$X_{\alpha+1} = K(X_\alpha), \quad X_\alpha = \bigcup\{X_\beta : \beta < \alpha\} \quad \text{for } \alpha \text{ limit},$$

and $X_{\beta+2} = K(A(X_{\beta+1}, X_\beta))$ in the other cases. It is easy to observe that X_κ is κ-superuniversal.

Katětov's maps are added in such a way that for every $y \in K(X) \setminus X$ there exists $Z \in [X]^{<\kappa}$ such that for all $x \in X$:

$$d(y,x) = \inf\{d(y,z) + d(z,x) : z \in Z\},$$

which allows us to show that $\mathcal{D}_w(x, X) \subseteq \mathcal{D}_w(x, K(X))$.

Reduction of a weak middle point

Assume that $D \notin \mathcal{D}_w(x, K(X))$. There exists a weak middle point $y \in K(X) \setminus X$ of some $D' \in [D]^\kappa$. Then for all $t \in D'$ there is $z \in Z$ such that

$$d(y, t) \leqslant d(y, z) + d(z, t) < 1.$$

Using the fact that $\kappa = \operatorname{cf} \kappa > |Z|$ we obtain $D'' \in [D']^\kappa$ and $z \in Z$ such that $d(y, z) + d(z, t) < 1$ for all $t \in D''$.
In particular $d(z, t) < 1$ for every $t \in D''$.
Once again, using $\kappa = \operatorname{cf} \kappa > \mathfrak{c}$, we can assume that

$$d(z, t) = d(z, t') < 1$$

for all $t, t' \in D''$. Thus $z \in Z \subseteq X$ is a weak middle point of D'', hence $D \notin \mathcal{D}_w(x, X)$.

Removing unwanted unitary subspaces

Fix a unitary subspace $D \subseteq X$. It suffices to choose some $D' \in [D]^\kappa$ and add a weak middle point y of D'. Then D will not be considered in the computing of $\tau_w(x, X \cup \{y\})$ for all $x \in D$, i.e. $D \notin \mathcal{D}_w(x, X \cup \{y\})$. We can do that for all the unwanted unitary subspaces at the same time.

Observe that if we want to remove a unitary subspace D' and $\{D_\alpha : \alpha < \lambda\}$ is a family of unitary subspaces we want to preserve, then D' has to satisfy

$$|D' \cap D_\alpha| < \kappa \quad \text{for all } \alpha < \lambda.$$

Unfortunately, it is not sufficient: if $D' = \{y_\beta : \beta < \kappa\}$, $\{x_\beta : \beta < \kappa\} \subseteq D_\alpha$ for some $\alpha < \lambda$, and $d(x_\beta, y_\beta) < \frac{1}{2}$, then adding x such that $d(x, y_\beta) = \frac{1}{2}$ it is difficult to ensure that x is not a weak middle point of $\{x_\beta : \beta < \kappa\}$.

Unitary character of a point

There is a special kind of a weak middle point: if $D \subseteq X$ is a unitary subspace then we say that $x \in X$ is a *middle point of D* if $d(x, y) = \frac{1}{2}$ for all $y \in D$. Analogously we define

$$\tau(x, X) = \sup\{|D| : D \in \mathcal{D}(x, X)\}$$

where

$$\mathcal{D}(x, X) = \{D \subseteq X : D \text{ is unitary}, x \in D \text{ and there is no middle point of any } D' \in [D]^\kappa\},$$

and this definition of unitary character has been used in the proof of the rigidity.

The cardinality of the example

If we define a sequence $(\mu_\alpha : \alpha \leqslant \kappa^+)$ by

$$\mu_0 = \kappa, \quad \mu_{\alpha+1} = (\aleph_{\mu_\alpha \cdot 3})^\kappa \text{ and } \mu_\beta = \sup\{\mu_\alpha : \alpha < \beta\}$$

for a limit ordinal β, then the cardinality of the example can be estimated by the number μ_{κ^+}, but its cofinality is κ^+.

Unitary character for an inaccessible cardinal

Let us assume that $\lambda > \kappa$ is an inaccessible cardinal, i.e. $\lambda = \mathrm{cf}\,\lambda$ and $2^\mu < \lambda$ for $\mu < \lambda$.
We can iterate operations of adding Katětov's maps, unitary subspaces (and removing unwanted unitary subspaces) λ many times. The resultant space will be rigid κ-superuniversal of cardinality λ.
This time we use the unitary character defined by the formula

$$\tau_<(x, X) = \sup\{|D| : D \in \mathcal{D}_<(x, X)\},$$

where

$$\mathcal{D}_<(x, X) = \{D \subseteq X : D \text{ is unitary}, |D| < |X|, x \in D \text{ and there is no middle point of any } D' \in [D]^\kappa\}.$$

S. H. Hechler, Large superuniversal metric spaces, Israel J. Math. 14(2) 1973, 115–148.

M. Katětov, On universal metric spaces, in: General Topology and its Relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Topological Symposium 1986, Z. Frolík (ed.), Berlin 1988.