# square sequences

Squares are very powerful combinatorial principles introduced by Jensen. They hold in $L$ for example.

The square principle for a cardinal $\kappa$, $\Box_\kappa$ says that there is a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \text{ limit}\}$ such that $C_\alpha$ is club in $\alpha$, $o.t.(C_\alpha) \leq \kappa$, and whenever $\beta \in C'_\alpha$ (that is a limit point of $C_\alpha$) then $C_\beta = C_\alpha \cap \beta$.

Jensen proved that if $V = L$ then $\Box_\kappa$ holds for all cardinals $\kappa$. To get the consistency of the negation of $\Box_\kappa$ you need a Mahlo, for a regular $\kappa$, and much more for singular $\kappa$.

## Theorem

*(Todorcevic) PID implies for all cardinals* $\kappa$ $\neg\Box_\kappa$.

Todorcevic proof is based on his analysis of walks, and so we begin with their definitions.

# coherent sequences

A club system on a limit ordinal $\lambda$ with uncountable cofinality is a sequence $C = \langle C_\alpha \mid \alpha \in \lambda \rangle$ such that for limit $\alpha < \lambda$ $C_\alpha$ is a club (closed unbounded) subset of $\alpha$, and $C_\alpha = \{\beta\}$ when $\alpha = \beta + 1$. We assume that 0 is always in $C_\alpha$.

Definitions:

1. $C$ is coherent if whenever $\alpha \in \lim C_\beta$ we have $C_\alpha = C_\beta \cap \alpha$.
2. $C$ is "threadable" (or trivial) iff it can be extended to a $\lambda + 1$ coherent system. That is, there is a club $C_\lambda$ in $\lambda$ such that for every $\delta \in \lim C_\lambda$, $C_\delta = C_\lambda \cap \delta$.
3. Jensen's $\square_\kappa$ sequence for a cardinal $\kappa$ is a coherent club sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ such that the order-type of each $C_\alpha$ is $\leq \kappa$.
4. Jensen's square $\square_\kappa$ is not threadable.

## Theorem (Todorcevic)

*PID implies that every coherent club system on an ordinal of uncountable cofinality is threadable. So there are no square $\square_\kappa$ sequences. (Jensen's square $\square_\kappa$ is not threadable.)*

# Definition of walks

Let $\langle C_\alpha \mid \alpha < \lambda \rangle$ be a club system on an ordinal $\lambda$ with uncountable cofinality. For every $\alpha \leq \beta < \lambda$ we shall define $walk(\alpha, \beta) = \langle \beta_0, \ldots, \beta_{n-1}$, and then define $\rho_2(\alpha, \beta) = n - 1$, by induction on $\beta$.

$$walk(\alpha, \alpha) = \langle \alpha \rangle$$

Correspondingly $\rho(\alpha, \alpha) = 0$. For $\beta > \alpha$ we define:

$$walk(\alpha, \beta) = \langle \beta \rangle^\frown walk(\alpha, \min(C_\beta \setminus \alpha)).$$

Correspondingly $\rho(\alpha, \beta) = 1 + \rho(\alpha, \min(C_\beta \setminus \alpha))$.

# Finite differences

## Lemma

*If the club system* $\langle C_\alpha \mid \alpha < \lambda \rangle$ *is coherent, then for every* $\alpha < \beta < \lambda$

$$\sup_{\xi < \alpha} |\rho(\xi, \alpha) - \rho(\xi, \beta)| < \infty. \tag{1}$$

Proof: by induction on $\beta$.

# The ideal $I_C$ of a coherent sequence $C$

Let $C = \langle C_\alpha \mid \alpha \in \lambda \rangle$ (with $\lambda$ of uncountable cofinality) be a coherent club system,

## Definition

$X \in I_C$ iff $X \subset \lambda$ is either finite or countable infinite and for some $\beta \geq \sup X$ we have that $\lim_{x \in X} \rho(x, \beta) = \infty$ (by this we mean that for every $n \in \omega$, for all but finitely many $x \in X$ we have $\rho(x, \beta) > n$.)

The Finite Difference Lemma implies that if $X \in I_C$ then actually for *every* $\beta \geq \sup X$ we have that $\rho(x, \beta)$ tends to infinity as $x \in X$.

## Lemma

*When cf $\lambda > \aleph_0$, $I_C$ is a $P$-ideal.*

Proof. Suppose $A_i \in I_C$ for $i \in \omega$.

1. Find one $\beta$ so that $\lim_{x \in A_i} \rho(x, \beta) = \infty$.

## Lemma

*When* $cf\,\lambda > \aleph_0$, $I_C$ *is a P-ideal.*

Proof. Suppose $A_i \in I_C$ for $i \in \omega$.

1. Find one $\beta$ so that $\lim_{x \in A_i} \rho(x, \beta) = \infty$.
2. Define $A_i' = A_i \setminus \{x \in A_i \mid \rho(x, \beta) \leq i\}$.

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2. Define $A_i' = A_i \setminus \{x \in A_i \mid \rho(x, \beta) \leq i\}$.
3. Then $A = \bigcup_i A_i' \in I_C$.

Recall the PID statement.

*If $I$ is a $P$-ideal over a set $S$, then either*
*$S$ is a countable union of sets that are out of $I$ (i.e. orthogonal to $I$), or else*
*$S$ contains an uncountable set that is inside $I$.*

## Lemma

*If $C$ is a coherent club system over $\lambda$ with cofinality $> \omega_1$, then the second alternative cannot hold for $I_C$. Namely there is no uncountable set inside $I_C$.*

Proof. Say $X$ is inside $I_C$. And of order-type $\omega_1$. As $cf(\lambda) > \omega_1$, can pick $\beta > \sup X$. But then there is some $n$ and an infinite $X_0 \subset X$ such that $\rho(x, \beta) = n$ for all $x \in X_0$. Thus $X_0 \notin I_C$.

# PID implies every club system is threadable

## Theorem (Todorcevic)

*Assume the PID. Assume $cf(\lambda) > \omega_1$. Every coherent club system over $\lambda$ is threadable.*

Proof. Consider the $P$-ideal $I_C$. The second alternative of the dichotomy does not hold. So $\lambda$ is a countable union of sets out of $I_C$. So there is a set $A \subset \lambda$ that is cofinal in $\lambda$ and is out of $A$. (No infinite subset of $A$ is in $I_C$.)

# PID implies every club system is threadable

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So for every $\beta < \lambda$ there is $n(\beta) \in \omega$ such that if $\alpha \in A \cap \beta$ then $\rho(\alpha, \beta) \leq n(\beta)$.

# PID implies every club system is threadable

**Theorem (Todorcevic)**

*Assume the PID. Assume $cf(\lambda) > \omega_1$. Every coherent club system over $\lambda$ is threadable.*

Proof. Consider the $P$-ideal $I_C$. The second alternative of the dichotomy does not hold. So $\lambda$ is a countable union of sets out of $I_C$. So there is a set $A \subset \lambda$ that is cofinal in $\lambda$ and is out of $A$. (No infinite subset of $A$ is in $I_C$.)
So for every $\beta < \lambda$ there is $n(\beta) \in \omega$ such that if $\alpha \in A \cap \beta$ then $\rho(\alpha, \beta) \leq n(\beta)$.
So there is $n$ such that $n = n(\beta)$ for an unbounded set of $\beta$s in $\lambda$. Let $n$ be minimal with this property.

# the proof continues

So we are at the following situation. $n$ is minimal so that there is an unbounded set $B \subset \lambda$ such that for every $\beta \in B\, \forall \alpha \in A \cap \beta\; \rho(\alpha, \beta) \leq n$.

Say $\alpha < \lambda$ is em good if $\alpha \in S^{\lambda}_{\aleph_0}$ $(cf(\alpha) = \omega)$ and for all $\beta \in B$ above $\alpha$, $\alpha \in C'_{\beta}$. Let $G \subset \lambda$ be the set of good points.

1. There is an unbounded in $\lambda$ set of good points.
2. Item 1 implies that the club system is threadable.

Let's check item 2. The point is that if $\alpha_1 < \alpha_2$ are good, then $C_{\alpha_1}$ is an initial segment of $C_{\alpha_2}$ and hence $\bigcup_{\alpha \textit{ is good}} C_{\alpha}$ is a club of $\lambda$ that threads the system $C$.

Here is an application due to Todorcevic of the Symmetric Dichotomy theorem.

**Theorem**

*PFA implies that there no S-spaces. In fact, the simple dichotomy for* $\aleph_1$*-generated ideals implies that there are no S-spaces.*

Proof. Recall the definition: An S-space is a regular, hereditarily separable, but not hereditarily Lindelof topological space. To prove that no such space exists (under the dichotomy), suppose that X is a regular topological space which is not hereditarily Lindelof and we shall prove that X is not hereditarily separable. Since X is not hereditarily Lindelof, X has a subspace $S = \{x_\alpha \mid \alpha < \omega_1\}$ such that every initial part $S_\delta = \{x_\alpha \mid \alpha \leq \delta\}$ is open in S (i.e. S is "right-separated"). We consider the subspace topology on S and shall find a subset of S which is not separable. Since S is regular, each $x_\alpha$ has an open neighborhood $U_\alpha$ with closure $\overline{U}_\alpha \subset S_\alpha$. These countable closed sets generate an ideal $I$. By the dichotomy, there is an uncountable set $D \subset S$ which is either "inside" or "out" of $I$. If D is in, then every countable subset $E$ of D is in $I$, which means that it is covered by a countable closed set, and hence $E$ is not dense in D. If D is out of $I$, then D has a finite intersection with every set in $I$. So in particular the intersection of $D$ with every $U_\alpha$ is finite. As $S$ is a Hausdorff space, $D$ is discrete (and therefore not separable). □