Definition: I is an ideal of countable sets over S if

1. $I \subset [S]^{\leq\omega}$
2. I is closed under subsets and finite unions.
3. It is convenient to assume that $[S]^{<\omega} \subseteq I$.
4. For simplicity, in these lectures, we assume $S = \omega_1$

If $X \subset S$, then $I \restriction X$ is the ideal restricted to subsets of X:

$$I \restriction X = \mathcal{P}(S) \cap I.$$

We say $X \subset S$ is trivial for I if

1. $I \restriction X = [X]^{<\omega}$, we then say X is “out of I” (orthogonal to I), or
2. $I \restriction X = [X]^{\leq\omega}$, we then say X is “inside I”.

Dichotomy for a family of ideals

The SIMPLE form of the dichotomy for a family of ideals of countable sets over ω_1 is the following statement: For every ideal I in the family

there is an uncountable $X \subseteq S$ which is trivial.

This is a Ramsey type statement.
In other words, there is an uncountable $X \subseteq \omega_1$ such that:

1. X is inside I, or
2. X is out of I.

Our aim

We shall prove the consistency of dichotomies for two families of ideals: ω_1-generated ideals, and *P*-ideals. In fact we will prove that such dichotomies are consequence of the PFA. (For ω_1-generated ideals this is work of Todorcevic. For *P*-ideals, Todorcevic and Abraham.)

We will give an application of PID (due to Todorcevic) that it implies $\mathfrak{b} \leq \omega_2$. (It is open whether it implies $\mathfrak{c} \leq \omega_2$).

For a cardinal κ, $H(\kappa)$ is the collection of all sets whose transitive closure has cardinality $< \kappa$.
$(H(\kappa), \in)$ is the structure whose universe is $H(\kappa)$ with the membership relation. It is useful to add a well-ordering $<$ of that universe, and so when we say $H(\kappa)$ we refer to the structure $(H(\kappa), \in, <)$.

Forcing a set out of I

Theorem: Let I be an ideal of countable subsets of ω_1 that is ω_1-generated. There is a proper poset P that forces an uncountable subset X out of I. This works (the generic X is uncountable) under the assumption that there is no uncountable subset of ω_1 that is inside of I.

Definition of P

Suppose I is generated by $\{A_\alpha \mid \alpha < \omega_1\}$ where $A_\alpha \subseteq \alpha$.
$p \in P$ iff $p = (x_p, d_p, N^p)$ is such that:

1. $N^p = \{N^p_0, \dots, N^p_{k-1}\}$ is a finite set of countable elementary substructures of $H(\aleph_2)$, $N^p_i \in N^p_{i+1}$. $I \in N^p_0$.
2. $x_p \in [\omega_1]^{<\omega}$ is "separated" by N^p: Say $x_p = \alpha_0 < \cdots < \alpha_k$, then we have $\alpha_0 < N^p_0 \cap \omega_1 < \alpha_1 < N^p_1 \cap \omega_1 \cdots N^p_{k-1} \cap \omega_1 < \alpha_k$.
3. For every α in x_p and structure N^p_i not containing α (α is "above N_i"): $\alpha \notin \bigcup\{X \mid X \in N^p_i$ is inside $I\}$.
4. $d_p \in [\omega_1]^{<\omega}$.

Define $q \leq p$ (q is more informative) iff

1. $x_p \subseteq x_q$, $d_p \subseteq d_q$, and $N^p \subseteq N^q$.
2. For every $\alpha \in d_p$, $x_p \cap A_\alpha = x_q \cap A_\alpha$.

The two main properties of P are:

1. P is “proper” so that it does not collapse ω_1
2. P generically adds an uncountable subset of ω_1 that is out of I, IF I CONTAINS NO UNCOUNTABLE SET INSIDE I.

Main idea

Lemma. Suppose I is an ideal of countable subsets of ω_1 and ω_1 is not inside I. If $M \prec H(\aleph_1)$ is countable and $I \in M$, then $M \cap \omega_1 \notin I$.

Proof. M “knows” that ω_1 is not inside I, and hence there is some $Y \in M$ so that $M \models$ “Y is countable and $Y \notin I$”.
Hence indeed Y is countable not in I.
There is an enumeration of Y, $Y = \{y_i \mid i \in \omega\}$.
So there is such an enumeration in M. But as $\omega \subset M$, each y_i is in M.
So $Y \subset M \cap \omega_1$. As $Y \notin I$, surely $M \cap \omega_1 \notin I$.

P-ideals

Definition: An ideal I of countable subsets is a P-ideal if whenever $\{A_i \mid i < \omega\} \subset I$, then there is some $A \in I$ so that $A_i \subseteq^* A$ for every $i \in \omega$.

Definition: The P-ideal dicotomy (PID): If I is an ideal of countable subsets of S (S of any cardinality) then either:

- S is the union of countably many sets that are out of I (orthogonal), or
- there is an uncountable set inside I.

Consistency of the PID

The PID is a consequence of the Proper Forcing Axiom. In fact it is also consistent with CH.
Exercise: the PID implies there are no Souslin trees (Abraham, Todorcevic). So Souslin hypothesis is consistent with CH (Jensen).

We will describe the proof of the following Theorem:

> *Given a P-ideal I over a set S, if S is not a countable union of sets that are orthogonal to I, then there exists a proper forcing notion P that introduces an uncountable subset inside I.*

Definition of P

Definition: $K \subseteq I$ is cofinal in I if it is cofinal in the almost inclusion ordering $\subseteq^*$. That is, for every $X \in I$ there is $Y \in K$ such that $X \subseteq^* Y$.

Let P be the poset of all pairs $p = (a_p, H_p)$ where $a_p \in I$ (so a_p is countable), and H_p is a countable collection of cofinal subsets of I.

Define $q \leq p$ iff $a_p \subseteq a_q$, $H_p \subseteq H_q$, and the following condition holds. For every $K \in H_p$, if $e = a_q \setminus a_p$ then

$$\{X \in K \mid e \subseteq X\} \in H_q.$$

Main idea: Adjoin your enemy into your court

The first idea for a poset that introduces an uncountable set inside I is to force with $(I, \subset)$. given a countable $M \prec H(\kappa)$ and condition $a \in I \cap M$ define an increasing sequence $a_i \in I \cap M$ that successively enters each of the dense sets in M.

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Here we use the fact that I is a P-ideal. Pick some $E \in I$ so that for every $A \in I \cap M$ $A \subset^* E$.
Then when we define a_{i+1} we require not only $a_i \subset a_{i+1}$ but also $a_{i+1} \setminus a_i \subset E$.

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New Problem: Perhaps for some dense set $D \in M$, if $b \in D \cap M$ is any condition such that $a = a_i \subset b$, then it is not the case that $b \subset E$ (the finite set $b \setminus E$ is non-empty). Solution: Declare E a “bad set”; let H be the collection of all bad sets. Then $H \in M$. Redefine your poset so that the pair (a, H) is a condition. Request that any extension of this condition is of the form $(b, H, \ldots)$ so that $a \subseteq b$ AND $b \setminus a \in h$ for some $h \in H$.