WS ON ABSTRACT ANALYSIS 2014

HOMOMORPHISMS, STRUCTURAL RAMSEY THEORY & LIMITS

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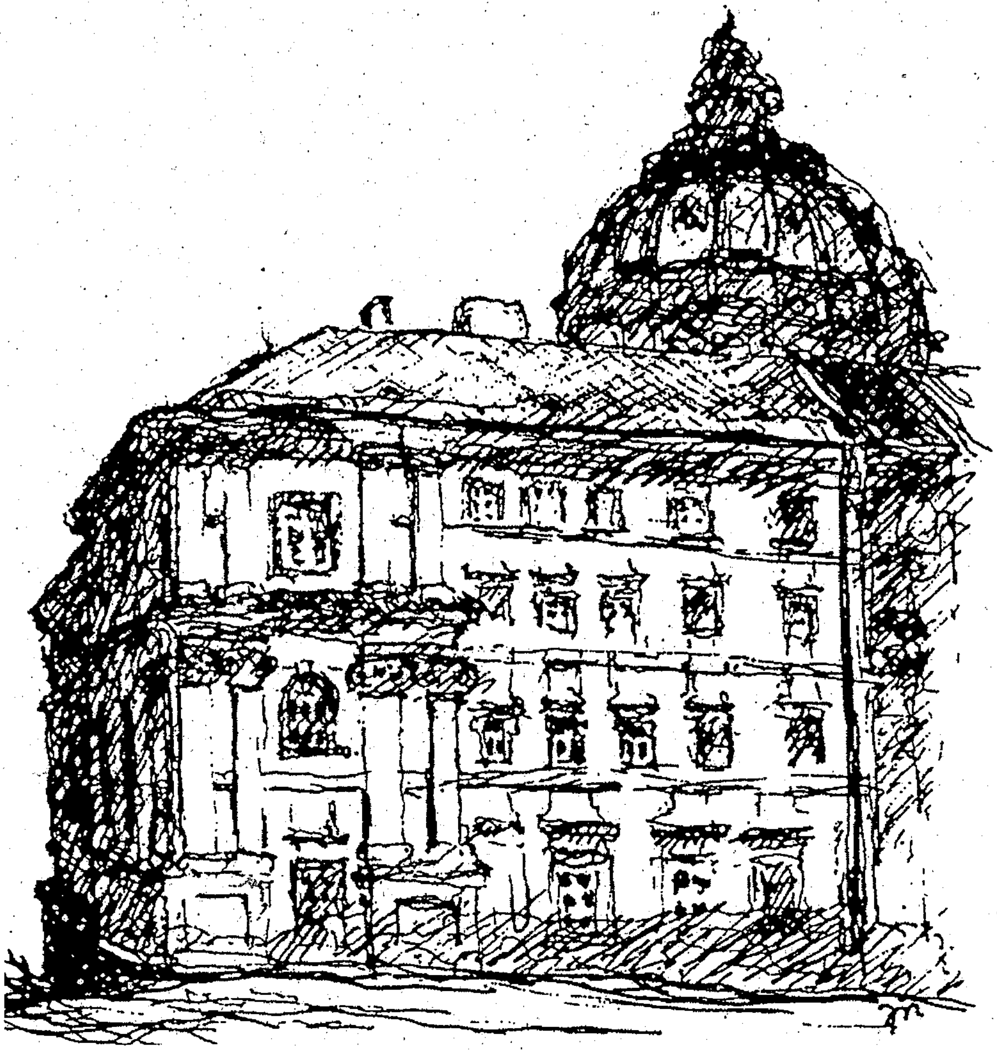


FINITE RAMSEY THEOREM

$\boxed{\text{FRT}}$

$$\forall p,k,n \ \exists N : \quad N \longrightarrow (n)^p_k$$

WHERE $N \longrightarrow (n)^p_k$ MEANS

FOR EVERY PARTITION

$\binom{N}{p} = a_1 \cup \ldots \cup a_k$ THERE EXISTS

$Y \in \binom{N}{n}$ AND i_0 SUCH THAT

$$\binom{Y}{p} \subseteq a_{i_0}$$

$N = \{0,1,\ldots,N-1\}$

$a_1 \cup \ldots \cup a_k$ ... COLORING

Y HOMOGENEOUS $\quad \binom{Y}{p} = \{A : A \subseteq Y \ \& \ |A| = p\}$

OTHER EXAMPLES:

RAMSEY THEOREM FOR PARTITION OF SUBCUBES
($\equiv$ PARAMETER SETS)

$\cup\!\!|$

HALES-JEWETT

RAMSEY THEOREM FOR PARTITIONS OF VECTOR SUBSPACES

RAMSEY THEOREM FOR PARTITIONS OF ORDERED SUBGRAPHS

$\vdots$

$\Downarrow$

GENERAL SETTING

$\boxed{\text{DEF}}$ (LEEB, N., RÖDL)

$\mathcal{C}$ A CLASS OF STRUCTURES WITH ISOMORPHISM & SUBSTRUCTURES.

$\binom{B}{A} = \{$ ALL SUBSTRUCTURES OF B ISOMORPHIC TO A $\}$

$\boxed{\text{PARTITION ARROW}}$ $C \longrightarrow (B)^A_k$

FOR EVERY PARTITION $\binom{C}{A} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$

THERE EXISTS $B' \in \binom{C}{B}$ AND i_0

SUCH THAT $\binom{B'}{A} \subseteq \mathcal{A}_{i_0}$.

B' MONOCHROMATIC COPY OF B
HOMOGENEOUS

$\binom{B}{A}$ COPIES OF A IN B

$\boxed{\text{DEF}}$

$\mathcal{C}$ HAS $\boxed{\text{A- RAMSEY PROPERTY}}$

IF FOR EVERY $B \in \mathcal{C}$ THERE EXISTS

$C \in \mathcal{C}$ SUCH THAT $C \longrightarrow (B)^A_2$.

$\mathcal{C}$ IS A $\boxed{\text{RAMSEY CLASS}}$

IF

$\forall A \, \forall B \, \exists C \qquad C \longrightarrow (B)^A_2$.

RAMSEY CLASS $\equiv$ A- RAMSEY FOR EVERY $A \in \mathcal{C}$

$\boxed{\text{CLASSICAL EXAMPLES}}$ OF RAMSEY CLASSES

FRT

(1) FINITE LIN. ORDERED SETS + MONOTONNE INJECTIONS

(1') FINITE SETS WITH INCLUSIONS

SIMPLICIAL CATEGORY

(2) $\Delta = (\delta_i : i \in I)$ $\delta_i \geq 1$

OBJECTS

$(X, (R_i : i \in I))$ X FINITE ORDERED SET

$R_i \subseteq X^{\delta_i}$

+

MONOTONNE EMBEDDINGS

REL(Δ) RELATIONAL STRUCTURES OF TYPE Δ

ABRAMSON, HARRINGTON ; N., RÖDL

(3) PARAMETER SETS GRAHAM, ROTHSCHILD

(4) FINITE VECTOR SPACES (OVER $\mathbb{F}$)
LEEB, GRAHAM, ROTHSCHILD

(5) (m, p, c)-SETS LEEB

MORE RECENT RAMSEY CLASSES

– $\boxed{\text{RESTRICTED RAMSEY CLASSES}}$

(5) THE CLASS OF ALL ORDERED K_k - FREE GRAPHS IS RAMSEY FORB(K_k) N. RÖDL

(6) THE CLASS OF ALL ORDERED $\mathcal{F}$ - FREE SYSTEMS (IN REL(Δ)) IS RAMSEY
WHERE $\mathcal{F} = \{F_1, \dots, F_t\}$ IS A FINITE SET OF $\boxed{\text{IRREDUCIBLE}}$ SYSTEMS

$(X, (R_i))$ $\quad \forall x, y \; \exists i \; \exists r \in R_i \; (x, y \in r)$

(7) – ORDERED FINITE METRIC SPACES (+ ISOMETRIC EMBEDDINGS) N.

(8) – DUAL RAMSEY CLASSES N. RÖDL
CARLSON – SIMPSON
SOLECKI

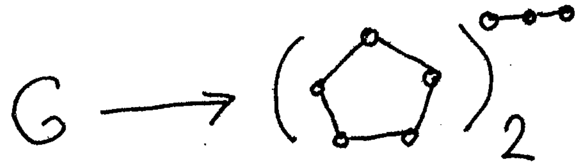
PROOFS

- AD HOC FOR CONCRETE $\mathcal{A}$-RAMSEY
- REPRESENTATION (SET –
  TYPE –)
- RANDOM (RÖDL, RUCZINSKI, SCHACHT)
- (COMPLICATED) INDUCTION
  (EARLY PAPERS
  "PASCAL THEORY")
- AMALGAMATION CONSTRUCTION
  "PARTITE"
  N. RÖDL

$\boxed{\text{OBSTACLES}}$ TO RAMSEY CLASSES

— RIGIDITY NEEDED

NO GRAPH G SATISFIES



— AMALGAMATION NEEDED

$\boxed{\text{KEY IDEA}}$

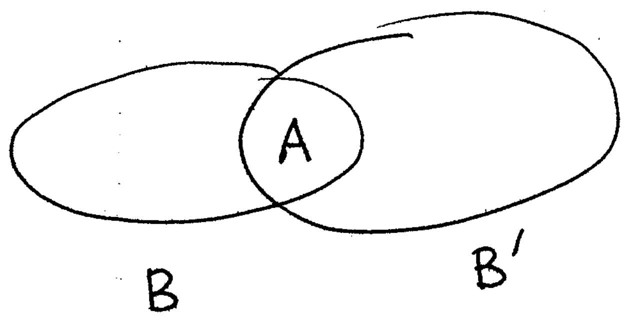
$$C \longrightarrow (B)^A_2 \iff \chi(X, \mathcal{M}) > 2$$

$$X = \binom{C}{A}$$

$$\mathcal{M} = \left\{ \binom{B'}{A} : B' \in \binom{C}{B} \right\}$$

$$\chi(X, \mathcal{M}) > 2 \Rightarrow \exists\, M, M' \in \mathcal{M}$$

$$|M \cap M'| = 1$$



(RAMSEY: A, B, B' ARBITRARY)

RAMSEY CLASS $\mathcal{C}$

$\Downarrow$

AMALGAMATION CLASS

$\Downarrow$

$\mathcal{C}$ AGE OF ULTRAHOMOGENEOUS STRUCTURE

RAMSEY CLASSES OF UNDIRECTED GRAPHS CHARACTERIZED

N. 89

CHARACTERIZATION PROGRAM

NEW CONTEXT OF RAMSEY CLASSES

KECHRIS, PESTOV, TODORCEVIC 05

CLASSIFICATION OF RAMSEY CLASSES BY MEANS OF HOMOMORPHISM

$$A = (X, (R_i : i \in I))$$
$$B = (X', (R'_i : i \in I))$$

$A \longrightarrow B$ IS A MAP $f : X \longrightarrow X'$

SATISFYING

$$(x_1, \ldots, x_{\delta_i}) \in R_i \Rightarrow (f(x_1), \ldots, f(x_{\delta_i})) \in R'_i$$

BASIC CLASSES

$\mathcal{F}$ A FINITE SET OF STRUCTURES

$$\mathcal{F} \longrightarrow \;=\; \{A : F \longrightarrow A \text{ FOR SOME } F \in \mathcal{F}\}$$

$$\mathcal{F} \not\longrightarrow \;=\; \{A : F \not\longrightarrow A \text{ FOR ALL } F \in \mathcal{F}\}$$

$$\longrightarrow D = \{A : A \longrightarrow D\}$$

$$\not\longrightarrow D = \{A : A \not\longrightarrow D\}$$

$$\longrightarrow D = \mathrm{CSP}(D)$$

$$\mathcal{F} \not\longrightarrow = \mathrm{FORB}(\mathcal{F})$$

$\boxed{\text{WHICH OF BASIC CLASSES ARE RAMSEY}}$?

FOR $\mathcal{S} \not\to$ AND $\longrightarrow D$

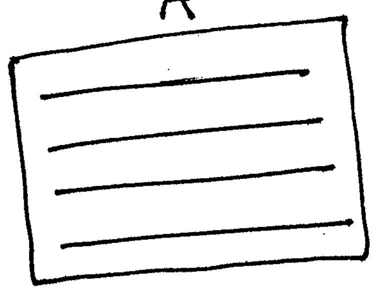
JUST IN A $\underbrace{\text{FEW CASES}}_{\text{ISOLATED}}$

HOWEVER IF WE CONSIDER $A \longrightarrow D$

AS EXTENSION A' OF A THEN IT IS
(LIFT) RAMSEY

$A \longrightarrow D$

A'

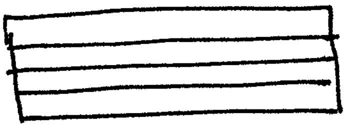


$\longrightarrow D$

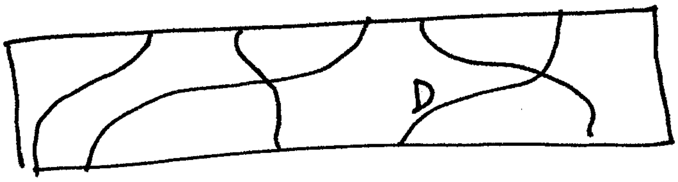
$\boxed{\text{PARTITE LEMMA}}$

FOR EVERY D THE EXTENSION CLASS $(\longrightarrow D)'$ HAS D-RAMSEY PROPERTY

FOR EVERY B'  $\longrightarrow D$

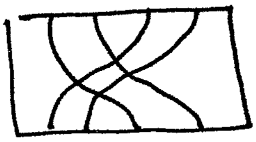
THERE EXISTS C'  $\longrightarrow D$

SUCH THAT $C' \longrightarrow (B')_2^{D'}$.



C'

$\cup\!|$



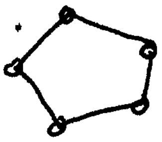
B'

HALES – JEWETT RELEVANCE

$\mathrm{FORB}(\mathcal{F}) = \mathcal{F} \not\to$

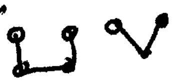
MOSTLY NOT RAMSEY

EXAMPLE

$\mathcal{F} = \{$  $\}$

$\mathcal{F} \not\to$ HAS NO  – RAMSEY PROPERTY

THERE IS NO $C \in \mathcal{F} \not\to$

SUCH THAT $C \longrightarrow ($  $)_2$ 

$\boxed{\text{THEOREM}}$

FOR EVERY FINITE SET $\mathfrak{F} \subseteq \mathrm{REL}(\Delta)$
THE CLASS $\mathrm{FORB}(\mathfrak{F})$ HAS AN EXTENSION
$\mathrm{FORB}(\mathfrak{F})' = \mathcal{C}$ WHICH IS RAMSEY.

$\boxed{\text{COROLLARY}}$

$\mathfrak{F}$ FIRST ORDER DEFINABLE HOM-CLOSED
THEN $\mathfrak{F} \not\rightarrow$ HAS A (FINITE) EXTENSION
WHICH IS RAMSEY

(THEOREM + ROSSMAN THEOREM)

$\boxed{\text{PROOF :}}$

AMALGAMATION CONSTRUCTION

INDUCTION ON $|\overline{\mathfrak{F}}|$

$$\overline{\mathfrak{F}} = \{\cancel{A} : F \underset{\text{EPI}}{\twoheadrightarrow} A \text{ FOR SOME } F \in \mathfrak{F}\}$$

FORB ($\mathcal{F}$) NOT AGE OF ULTRAHOMOGENEOUS

BUT AGE OF $\underbrace{\text{UNIVERSAL}}_{\text{(COUNTABLY)}}$

$\boxed{\text{THM}}$ (COROLLARY OF CHERLIN, SHELAH, SHE)

FOR EVERY FINITE $\mathcal{F}$

$\mathcal{F} \not\to$ HAS UNIVERSAL

CHERLIN, SHELAH, SHE MODEL TH.

COVINGTON

HUBIČKA, N. EXPLICIT AMALGAMATION ("FORBIDDEN PIECES")

SOMETIMES

$\mathcal{F} \not\to \; = \; \longrightarrow D$

FORB $(\mathcal{F})$ = CSP(D) (HOMO-DUALITY)

TWO VERY DIFFERENT RAMSEY (LIFTS) EXTENSIONS

? CHARACTERIZATION ?

(DUALITY :
C. TARDIF
P. OSSONA DE MENDEZ
CHARACTERIZATION)

OPEN PROBLEMS

- χARACTERIZATION OF RAMSEY CLASSES

- χARACTERIZATION OF UNIVERSALS WITH A RAMSEY EXTENSION

- χARACTERIZATION OF ULTRAHOMOGENEOU WITH A RAMSEY EXTENSION

- CAN ONE PROVE ALL RAMSEY CLASSES BY A VARIANT OF AMALGAMATION CONSTRUCTION?