WS on Abstract Analysis 2014

Homomorphisms, Structural Ramsey Theory & Limits

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HOMOMORPHISMS
\&
LOGIC
\&
LIMITS

(PARTICULARLY FOR SPARSE STRUCTURES)
ALL STRUCTURES IN A NOWHERE DENSE CLASS

— MAY BE FINITELY APPROXIMATED
  (WITH ARBITRARY PRECISION)

— SATISFY HOMOMORPHISM PRESERVATION
  (ROSSMAN; DAWAR; NPOM)

— RELATE TO DESCRIPTIVE COMPLEXITY
HOMOMORPHISM $G \rightarrow H$ is a mapping $f: V(G) \rightarrow V(H)$ satisfying:

$$xy \in E(G) \Rightarrow f(x)f(y) \in E(H).$$

"EDGE PRESERVING MAPPINGS"

(NOT ONLY GRAPHS; FINITE RELATIONAL SYSTEMS)
Homomorphism $G \to H$ is a mapping $f : V(G) \to V(H)$ satisfying:

$$xy \in E(G) \Rightarrow f(x)f(y) \in E(H).$$

"Edge preserving mappings"

(Not only graphs; finite relational systems)

$f : G \to H \equiv H$-colouring

$G \to K_3 \equiv 3$-colouring

$\triangle$
When is $H$-colouring hard problem

**Thm** (Hell, N., 1990)

$H$-colouring hard $\iff$ $H$ non-bipartite

Other proofs:
- Bulatov
- Siggers
- Barto-Kozik
- Kun-Szegedy

All hard
H-critical graph $G$

III

$G \rightarrow H$

$G' \rightarrow H$

For every proper subgraph

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When are there finitely many H-critical graphs?

III

Do there exists $F_1, \ldots, F_t$ such that for every $G$

$F_1 \rightarrow G$

$F_2 \rightarrow G$

$\vdots$

$F_t \rightarrow G$

$\iff G \rightarrow H$
\[ H = \mathcal{D} = T_3 \]

The only \( H \)-critical is

\[ \rightarrow P_3 \]

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\[ \vec{P}_k \rightarrow G \iff G \rightarrow \vec{T}_k \]

Gallai, Hasse, Roy, Vитаев
INFINITELY MANY $H$-CRITICAL
\[ H = \begin{array}{c}
\end{array} \]

\[
\text{INFINITELY MANY } H\text{-CRITICAL}
\]

\[
H = \begin{array}{c}
\end{array} \]

\[
\text{INFINITELY MANY } H\text{-CRITICAL}
\]
H = \{ \}

INFINITELY MANY H-CRITICAL

H = \downarrow \rightarrow

INFINITELY MANY H-CRITICAL

H = \uparrow \rightarrow \rightarrow

THE ONLY H-CRITICAL

\uparrow \uparrow

KOMAŘEK
MORE FORMALISM

\[ F \rightarrow = \{ A : F \rightarrow A \} \]

\[ \text{FORB}(F) \]

\[ \text{FORB}(\mathcal{F}) \quad \text{FOR} \quad \mathcal{F} = \{ F_1, \ldots, F_\ell \} \]

**THM** ANY NP PROBLEM
POLYNOMIALLY EQUIVALENT TO
MEMBERSHIP \( \text{FORB}(\mathcal{F}') \) FOR
A LIFT (EXPANSION \( \mathcal{I}' \) OF \( \mathcal{I} \))

(KUN, N.)

**THM** ANY HOMOMORPHISM
CLOSED FIRST ORDER DEFINABLE CLASS
IS OF FORM \( \mathcal{F} \rightarrow \)

(ROSSMAN)
FINITE HOM-DUALITY \textit{(n., Pultr)}

\[ \text{FORB}(\mathcal{F}) = \text{CSP}(D) \]

(EQUATION OF CLASSES)
GAME OF ALTERNATIVES:
FOR EVERY \( G \) HOLDS:
\[ F \to G \quad \text{FOR ALL} \quad F \in \mathcal{F} \]
IFF
\[ G \to D \]

\[ \mathcal{F}, F \in \mathcal{F} \ldots \text{FORBIDDEN GRAPHS} \]
\[ D \ldots \text{DUAL} \]

JUST_EXPRESSES_THE_FACT
THAT THERE ARE FINITELY MANY
\( D \)-CRITICAL GRAPHS.
FINITE DUALITIES CHARACTERIZED

- COMBINATORICS
  \( S \) A SET OF TREES
  \( D^2 \) DISMANTABLE

- LOGIC
  (ONLY FO DEFINABLE CSP)

- HOMOMORPHISM
  POSET
    / \
  GAPS CETS BOUNDS

(HEYTING POSETS)

KOMAREK
N., TARDIF
LAROSE, LOTTEN, TARDIF

ATSERIAS
ROSSMAN
$G \rightarrow H$ may be viewed as quasiorder defined on the class of all finite graphs if we consider non-isomorphic core graphs then we get partial order $\nabla$ homomorphism order
SPECTACULAR PROPERTIES
OF HOMOMORPHISM ORDER

- DENSE
- UNIVERSAL
- $\infty$ CONNECTED
  CACTI-LIKE
  UNDIRECTED

UNDIRECTED

DIRECTED
$\mathcal{E}$ - Restricted Duality

$\mathcal{E}$ a class of graphs

$\mathcal{F} \subseteq \mathcal{E}$

$\text{FORB}(\mathcal{F}) \cap \mathcal{E} = \text{CSP}(D) \cap \mathcal{E}$
DUALITY
REstricted Duality
\( \mathcal{C} \) has all restricted dualities iff connected for every finite set \( S \subseteq \mathcal{C} \).

There exists \( D_S \) such that

\[ \mathcal{C} \cap \text{FORB}(S) = \mathcal{C} \cap \text{CSP}(D_S) \]

— BOUNDED DEGREE GRAPHS HAVE ALL RESTRICTED DUALITIES (HAGKVIST, HELL)

— PLANAR GRAPHS HAVE ARD (N., P. OSSONA DE MENDEZ)
PLANAR - RESTRICTED DUALITY
Which classes have ARD?

Characterization by

- Metric properties of homomorphism order
- Oriented and acyclic lifts
- By subdivisions
- By FO definability (modulo Erdős-Hajnal conjecture, weak)
\[ \text{dist}(A, B) = 2^{-k} \]

\[ k = \min \left\{ |C| : C \rightarrow A \text{ or } C \rightarrow B \right\} \]

\[ C \rightarrow A \text{ or } C \rightarrow B \]

\[ \epsilon > 0 \]

\[ \phi^\epsilon(A) = \min \left\{ |B| : A \rightarrow B \right\} \]

\[ \text{dist}(A, B) < \epsilon \]

**THM** (N, POM)

**FOR A CLASS C**

1. \( C \) HAS ALL RESTRICTED DUALITIES

\[ \sup \phi^\epsilon(A) < \infty \]

\( A \in C \)

(For every \( \epsilon > 0 \))
dist defines \( \mathcal{X} \) completion of the homomorphism order

DUALITIES IN \( \mathcal{X} \) characterized

\( (F, \overline{D}) \) duality in \( \mathcal{X} \)

\[ \Downarrow \]

either \( F \) is a connected graph

or \( \overline{D} \) is a multiplicative graph
**REMARK**  
**DICHOTOMY BY COUNTING**

\[
\frac{\log \text{hom}(F,G)}{\log |G|} \rightarrow \{-\infty, 0, 1, \ldots, \eta(F) |E|\}
\]

**CLASS LIMIT SUPREMA IN THE RESOLUTION**

"THE LIMIT LOG-DENSITY" $\leq |F|$

FOR **EVERY** NON-DISCRETE $F$

$\uparrow$

$\in$ NOWHERE DENSE

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**COMPARE CONVERGENCE BY**

LOVASZ, SZEGEDY, SOS, …
E STRUCTURAL LIMITS

FINITE RELATIONAL LANGUAGE
(GRAPHS, DIGRAPHS, \(\varepsilon\)-COLORED GRAPHS ...)

\(\text{FO}(\mathcal{X})\) ALL FIRST-ORDER FORMULAS
\(X \subseteq \text{FO}(\mathcal{X})\) A FRAGMENT

**DEFINITION**

\(G_1, G_2, \ldots, G_n, \ldots\) IS \(X\)-CONVERGENT

IF FOR EVERY \(\varphi \in X\) THE SEQUENCE

\(\langle \varphi_1 G_1 \rangle, \langle \varphi_1 G_2 \rangle, \ldots, \langle \varphi_1 G_n \rangle, \ldots\)

CONVERGES.

**HERE:**

\[
\langle \varphi_1 G_n \rangle = \frac{|\{(n_1, \ldots, n_p) \mid G_n = \varphi(n_1, \ldots, n_p)\}|}{|G_n|^p}
\]

\(\varphi = \varphi(x_1, \ldots, x_p)\) FORMULA WITH \(p\)
FREE VARIABLES
boolean algebra

boolean subalgebra

stone space of $X$

(ultrafilters or)

homomorphisms $f: X \rightarrow \beta X$

+ topology by

$K_x(\varphi) = \{ U | \varphi \in U \}$

$= \{ f | f(\varphi) = 1 \}$

stone duality
THEOREM

For every $X$-convergent sequence $(G_n)_{n \in \mathbb{N}}$ there exists unique probability measure $\mu$ on $S(x)$ such that for every formula $\varphi(x_1, \ldots, x_p) \in X$ holds:

$$\int_{S(x)} 1_{K(\varphi)}(x) \, d\mu(x) = \lim_{n \to \infty} \langle \varphi, G_n \rangle$$
SPECIAL FRAGMENTS

$QF(x)$ QUANTIFIER FREE

$FO_0(x)$ NO FREE VARIABLES (SENTENCES)

$FO_p(x)$ $p$-FREE VARIABLES

$FO^{local}(x)$ LOCAL FORMULAS
$F_0(\mathcal{L}) \xrightarrow{\uparrow} \text{CONVERGENCE}$

$\iff$ \text{ELEMENTARY CONVERGENCE}$

$(G_n)_{n \in \mathbb{N}}$ \text{ iff for every sentence } \varphi$

\text{ELEMENTARY CONV.}$

$G_n \models \varphi$

\text{ for all } n \geq n_0(\varphi)$. 
QF - CONVERGENCE

\[ \uparrow \]

L - CONVERGENCE

\((g_n)_{n \in \mathbb{N}}\) L - CONVERGENT

\[ \text{AFF} \]

\[ \text{hom}(F, G) \]

\[ |G| |F| \]

CONVERGENT

FOR EVERY F
Graphs with $\Delta \leq D$

BS - CONVERGENCE (Benjamini)
SCHRAMM

III

$FO_{local}^1$ - CONVERGENCE

(STATISTICS OF ISOMORPHISM TYPES OF NEIGHBORHOOD CONVERGES)

THM

$(G_n)_{n \in \mathbb{N}}$ with $\Delta(G_n) \leq D$

is $FO$ - CONVERGENT

$(G_n)$ is BS-CONVERGENT

$\forall$ $FO_0$ - CONVERGENT
BACK TO

SPARSE VS DENSE

DICHOTOMY

LOVASZ: LIMITS FOR "INTERMEDIATE" CLASSES?
For classes of graphs with bounded tree depth, one can describe limit objects. Hope for general bounded expansion via low-tree-depth decomposition. Central problem of sparse graphs.
Sparsity

Graphs, Structures, and Algorithms

Springer