A boolean algebraic approach to semiproper iterations II

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Generalized stationarity

Definition

Let X be an uncountable set. A set C is a *club* on $\mathcal{P}(X)$ iff there is a function $f_C : X^{<\omega} \to X$ such that C is the set of elements of $\mathcal{P}(X)$ closed under f_C, i.e.

$$C = \{Y \in \mathcal{P}(X) : f_C[Y]^{<\omega} \subseteq Y\}$$

A set S is *stationary* on $\mathcal{P}(X)$ iff it intersects every club on $\mathcal{P}(X)$.

Example

The set $\{X\}$ is always stationary since every club contains X. Also $\mathcal{P}(X) \setminus \{X\}$ and $[X]^\kappa$ are stationary for any $\kappa \leq |X|$ (following the proof of the well-known downwards Löwhenheim-Skolem Theorem). Notice that every element of a club C must contain $f_C(\emptyset)$, a fixed element of X.

Remark

The reference to the support set X for clubs or stationary sets may be omitted, since every set S can be club or stationary only on $\bigcup S$.

Given any first-order structure M, from the set M we can define a Skolem function $f_M : M^{<\omega} \to M$ (i.e., a function coding solutions for all existential first-order formulas over M). Then the set C of all elementary submodels of M contains a club (the one corresponding to f_M). Henceforth, every set S stationary on X must contain an elementary submodel of any first-order structure on X.

Definition

The *club filter* on X is

$$\mathrm{CF}_X = \{C \subset \mathcal{P}(X) : \ C \text{ contains a club}\}.$$

Similarly, the *non-stationary ideal* on X is

$$\mathrm{NS}_X = \{A \subset \mathcal{P}(X) : \ A \text{ not stationary}\}.$$

Lemma

CF_X *is a* σ*-complete filter on* $\mathcal{P}(X)$*, and the stationary sets are exactly the* CF_X*-positive sets.*

Definition

Given a family $\{S_a \subseteq \mathcal{P}(X) : \ a \in X\}$, the *diagonal union* of the family is $\nabla_{a \in X} S_a = \{z \in \mathcal{P}(X) : \ \exists a \in z \; z \in S_a\}$, and the *diagonal intersection* of the family is $\Delta_{a \in X} S_a = \{z \in \mathcal{P}(X) : \forall a \in z \; z \in S_a\}$.

Lemma (Fodor)

CF_X *is normal, i.e. is closed under diagonal intersection. Equivalently, every function* $f : \ \mathcal{P}(X) \to X$ *that is regressive on a* CF_X*-positive set is constant on a* CF_X*-positive set.*

From now on we shall be interested just in stationary subsets of $[X]^{\aleph_0}$ for suitable uncountable sets X.

(SEMI)PROPERNESS

Definition

Let $\mathbb{B}$ be a complete boolean algebra and $M \prec H_\theta$ be countable with $\theta \gg |\mathbb{B}|$.
$\text{PD}_{\aleph_1}(\mathbb{B})$ is the collection of predense subsets of $\mathbb{B}$ of size at most ω_1.
$\text{PD}(\mathbb{B})$ is the collection of predense subsets of $\mathbb{B}$.
The boolean value

$$sg(\mathbb{B}, M) = \bigwedge \left\{ \bigvee (D \cap M) : D \in \text{PD}_{\aleph_1}(\mathbb{B}) \cap M \right\}$$

is the *degree of semigenericity* of M with respect to $\mathbb{B}$.
The boolean value

$$gen(\mathbb{B}, M) = \bigwedge \left\{ \bigvee (D \cap M) : D \in \text{PD}(\mathbb{B}) \cap M \right\}$$

is the *degree of genericity* of M with respect to $\mathbb{B}$.

Proposition

Let $\mathbb{B}$ be a complete boolean algebra and $M \prec H_\theta$ for some $\theta \gg |\mathbb{B}|$. Then for all $b \in M \cap \mathbb{B}$

$$sg(\mathbb{B} \upharpoonright b, M) = sg(\mathbb{B}, M) \wedge b.$$

$$gen(\mathbb{B} \upharpoonright b, M) = gen(\mathbb{B}, M) \wedge b.$$

Definition

Let $\mathbb{B}$ be a complete boolean algebra.

- $\mathbb{B}$ is semiproper (SP) iff for club many $M \prec H_\theta$ in $[H_\theta]^{\aleph_0}$ whenever b is in $\mathbb{B}^+ \cap M$, we have that $sg(\mathbb{B}, M) \wedge b > 0_{\mathbb{B}}$.
- $\mathbb{B}$ is proper iff for club many $M \prec H_\theta$ in $[H_\theta]^{\aleph_0}$ whenever b is in $\mathbb{B}^+ \cap M$, we have that $gen(\mathbb{B}, M) \wedge b > 0_{\mathbb{B}}$.

Baire category theorem for Stone spaces

Let $\mathbb{B}$ be a complete boolean algebra, $X_{\mathbb{B}}$ be the Stone space of its ultrafilters,

$$N_a = \{G \in X_{\mathbb{B}} : a \in G\}.$$

Notice that A is a predense subset of $\mathbb{B}$ iff

$$X_A = \bigcup \{N_a : a \in A\}$$

is open dense in $X_{\mathbb{B}}$ (but in general not regular).

The Baire category theorem holds for $X_{\mathbb{B}}$: If $\{A_n : n \in \omega\}$ is a family of predense subsets of $\mathbb{B}$

$$X = \bigcap_{n \in \omega} X_{A_n}$$

is comeager in $X_{\mathbb{B}}$, thus

$$\overline{\mathring{X}} = X_{\mathbb{B}}.$$

Let $M \prec H_\theta$ be countable, $\mathbb{B} \in M$, then if $\{B_n : n \in \omega\}$ is the set of predense subsets of $\mathbb{B} \in M$, the classical construction of an M-generic filter shows that

$$\bigcap_{n\in\omega} \left(\bigcup \{N_a : a \in B_n \cap M\} \right) \neq \emptyset.$$

This does not guarantee that

$$\bigcap_{n\in\omega} \left(\bigcup \{N_a : a \in B_n \cap M\} \right) \text{ is comeager on some } N_b \text{ in } V.$$

Topological characterization of properness

Proposition

$\mathbb{B}$ is proper if and only if $\forall M \prec H_\theta$ with $\mathbb{B} \in M$, M countable

$$X_M = \bigcap \left\{ \bigcup \{N_a : a \in B \cap M\} : B \in M \text{ predense subset of } \mathbb{B} \right\}$$

is such that $\forall c \in M \cap \mathbb{B} \exists b \in \mathbb{B}$ such that X_M is comeager set on $N_b \cap N_c$.

Proof.

As a matter of fact

$$\forall c \in M \cap \mathbb{B} \; \exists b (N_b \subseteq \overline{\mathring{X}_M} \cap N_c)$$

$$\iff \forall c \in M \cap \mathbb{B} \exists b \leq \bigwedge \left\{ \bigvee (A \cap M) : A \in M \text{ maximal antichain} \right\} \wedge c.$$

□

Shelah's semiproperness

Definition

(Shelah) Let P be a partial order, and fix $M \prec H_\theta$. Then q is a *M-semigeneric* condition for P iff for every $\dot{\alpha} \in V^P \cap M$ such that $1_P \Vdash \dot{\alpha} < \check{\omega}_1$,

$$q \Vdash \dot{\alpha} < M \cap \omega_1.$$

P is semiproper in the sense of Shelah if there exists a club C of elementary substructures of H_θ such that for every countable $M \in C$, there exist a M-semigeneric condition below every element of $P \cap M$.

Proposition

Let $\mathbb{B}$ *be a complete boolean algebra, and fix* $M \prec H_\theta$*. Then*

$$sg(\mathbb{B}, M) = \bigvee \{q \in \mathbb{B} : \ q \text{ is a } M\text{-semigeneric condition}\}$$

Proposition

P *is semiproper in the sense of Shelah iff* $\mathrm{RO}(P)$ *is semiproper.*

Two-step iterations of semiproper posets

Recall:

Definition

Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, the *retraction* associated to i is the map

$$\begin{array}{rccc} \pi_i : & \mathbb{C} & \to & \mathbb{B} \\ & c & \mapsto & \bigwedge \{b \in \mathbb{B} : \; i(b) \geq c\} \end{array}$$

Proposition

Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $b \in \mathbb{B}$, $c, d \in \mathbb{C}$ be arbitrary. Then,

1. $\pi_i \circ i(b) = b$ *hence* π_i *is surjective;*
2. $i \circ \pi_i(c) \geq c$ *hence* π_i *maps* $\mathbb{C}^+$ *to* $\mathbb{B}^+$*;*
3. π_i *preserves joins, i.e.* $\pi_i(\bigvee X) = \bigvee \pi_i[X]$ *for all* $X \subseteq \mathbb{C}$*;*
4. $i(b) = \bigvee \{e : \pi_i(e) \leq b\}$.
5. $\pi_i(c \wedge i(b)) = \pi_i(c) \wedge b = \bigvee \{\pi_i(e) : e \leq c, \pi_i(e) \leq b\}$*;*

Definition

$i : \mathbb{B} \to \mathbb{C}$ is semiproper (SP) iff $\mathbb{B} \in$ SP and for club many $M \in [H_\theta]^{\aleph_0}$, whenever c is in $\mathbb{C}^+ \cap M$ we have that

$$\pi(c \wedge sg(\mathbb{C}, M)) = \pi(c) \wedge sg(\mathbb{B}, M).$$

Proposition

Let $\mathbb{B}$ *be a semiproper complete boolean algebra, and let* $\dot{\mathbb{C}}$ *be such that*

$$[\![\dot{\mathbb{C}} \in \mathsf{SP}]\!] = 1_{\mathbb{B}},$$

then $\mathbb{D} = \mathbb{B} * \dot{\mathbb{C}}$ *and* $i_{\mathbb{B}*\dot{\mathbb{C}}} : \mathbb{B} \to \mathbb{D}$ *are both semiproper.*

Lemma

Let $\mathbb{B}$, $\mathbb{C}_0$, $\mathbb{C}_1$ be semiproper complete boolean algebras, and let G be any V-generic filter for $\mathbb{B}$. Let i_0, i_1, j form a commutative diagram of regular embeddings as in the following picture:

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \\ & \searrow^{i_1} & \downarrow j \\ & & \mathbb{C}_1 \end{array}$$

Moreover assume that $\mathbb{C}_0/i_0[G]$ is semiproper in $V[G]$ and

$$\left[\!\!\left[\mathbb{C}_1/j[\dot{G}_{\mathbb{C}_0}] \text{ is semiproper} \right]\!\!\right]_{\mathbb{C}_0} = 1_{\mathbb{C}_0}.$$

Then in $V[G]$, $j/_G : \mathbb{C}_0/_G \to \mathbb{C}_1/_G$ is a semiproper embedding.

Recall:

Definition

Let $\mathcal{F}$ be a complete iteration system of length λ.

- The *inverse limit* of the iteration is

$$T(\mathcal{F}) = \left\{f \in \Pi_{\alpha<\lambda}\mathbb{B}_\alpha : \ \forall\alpha\forall\beta > \alpha \ \pi_{\alpha\beta}(f(\beta)) = f(\alpha)\right\}$$

  and its elements are called *threads*.

- The *direct limit* is

$$C(\mathcal{F}) = \left\{f \in T(\mathcal{F}) : \ \exists\alpha\forall\beta > \alpha \ f(\beta) = i_{\alpha\beta}(f(\alpha))\right\}$$

  and its elements are called *constant threads*. The support of a constant thread $\mathrm{supp}(f)$ is the least α such that $i_{\alpha\beta} \circ f(\alpha) = f(\beta)$ for all $\beta \geq \alpha$.

- The *revised countable support limit* is

$$RCS(\mathcal{F}) = \left\{f \in T(\mathcal{F}) : \ f \in C(\mathcal{F}) \vee \exists\alpha \ f(\alpha) \Vdash_{\mathbb{B}_\alpha} \mathrm{cf}(\check{\lambda}) = \check{\omega}\right\}$$

Lemma

Assume $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ *is such that*

$$[\![\mathbb{B}_{\alpha+1}/i[\dot{G}_{\mathbb{B}_\alpha}] \text{ is semiproper}]\!]_{\mathbb{B}_\alpha} = 1_{\mathbb{B}_\alpha}$$

for all $\alpha < \lambda$.
Let G_α *be* V*-generic for* $\mathbb{B}_\alpha$*. Then*

$$\mathcal{F}/G_\alpha = \{i_{\eta\beta}/_{G_\alpha} : \alpha \leq \eta \leq \beta < \lambda\}$$

is in $V[G_\alpha]$ *an iteration system made of semiproper embeddings.*

Definition

An iteration system $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ is semiproper iff $i_{\alpha\beta}$ is semiproper for all $\alpha \leq \beta < \lambda$.
An iteration system $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ is RCS iff $\mathbb{B}_\alpha = \mathrm{RO}(RCS(\mathcal{F} \upharpoonright \alpha))$ for all $\alpha < \lambda$.

Lemma

Let $\mathcal{F} = \{i_{nm} : n \leq m < \omega\}$ *be a semiproper iteration system. Then* $T(\mathcal{F})$ *and the corresponding* $i_{n\omega}$ *are also semiproper.*

Lemma

Let $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \omega_1 \right\}$ *be an* RCS *and semiproper iteration system. Then* $C(\mathcal{F})$ *and the corresponding* $i_{\alpha\omega_1}$ *are semiproper.*

Lemma

Let $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda \right\}$ *be an* RCS *and semiproper iteration system such that* $C(\mathcal{F})$ *is* $< \lambda$*-cc. Then* $C(\mathcal{F})$ *and the corresponding* $i_{\alpha\lambda}$ *are semiproper.*

Theorem

Let $\mathcal{F} = \left\{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\right\}$ *be an* RCS *iteration system, such that for all* $\alpha < \beta < \lambda$,

$$\left[\!\!\left[\mathbb{B}_\beta / i_{\alpha\beta}[\dot{G}_\alpha] \text{ is semiproper}\right]\!\!\right] = 1_{\mathbb{B}_\alpha}$$

and for all α *there is a* $\beta > \alpha$ *such that* $\mathbb{B}_\beta \Vdash |\mathbb{B}_\alpha| \leq \omega_1$. *Then* RCS$(\mathcal{F})$ *and the corresponding* $i_{\alpha\lambda}$ *are semiproper.*

Fact

Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \le \beta < \lambda\}$ *be a semiproper iteration system, f be in* $T(\mathcal{F})$*. Then*

$$\mathcal{F} \restriction f = \{(i_{\alpha\beta})_{f(\beta)} : \mathbb{B}_\alpha \restriction f(\alpha) \to \mathbb{B}_\beta \restriction f(\beta) : \alpha \le \beta < \lambda\}$$

is a semiproper iteration system and its associated retractions are the restriction of the original retractions.

Lemma

Let $\mathcal{F} = \left\{ i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda \right\}$ *be an* RCS *and semiproper iteration system. Let* $M \prec H_\theta$ *be countable,* $g \in M$ *be any condition in* $\text{RCS}(\mathcal{F})$*,* $\dot{\alpha} \in M$ *be a name for a countable ordinal,* $\delta \in M$ *be an ordinal smaller than* λ*.*
Then there exists a condition $g' \in \text{RCS}(\mathcal{F}) \cap M$ *below* g *with* $g'(\delta) = g(\delta)$ *and* $g' \wedge i_\delta(sg(\mathbb{B}_\delta, M))$ *forces that* $\dot{\alpha} < M \cap \omega_1$*. If* $\lambda = \omega_1$*, then the support of* $g' \wedge i_\delta(sg(\mathbb{B}_\delta, M))$ *is contained in* $M \cap \omega_1$*.*