Constructing special almost disjoint families

Dilip Raghavan

National University of Singapore

Winter School in Abstract Analysis,
International Center for Spiritual Rehabilitation, Hejnice
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Outline

1. A weakly tight family from $\mathfrak{s} \leq \mathfrak{b}$

Recall:

Definition

An a.d. $\mathscr{A} \subset [\omega]^\omega$ is weakly tight if for any collection $\{b_n : n \in \omega\} \subset \mathcal{I}^+(\mathscr{A})$, there exists $a \in \mathscr{A}$ such that $\exists^\infty n \in \omega\, [|a \cap a_n| = \omega]$.

- Recall that this is a weakening of $\aleph_0$-MAD, which in turn is essentially the same as Cohen-indestructible.

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- Recall that this is a weakening of $\aleph_0$-MAD, which in turn is essentially the same as Cohen-indestructible.
- One cannot directly apply Shelah's method to construct a weakly tight family. Why?

Definition

A partitioner of an a.d. family $\mathscr{A}$ is a set $b \in \mathcal{I}^{+}(\mathscr{A})$ with the property that $\forall a \in \mathscr{A} \, [a \subset^{*} b \vee |a \cap b| < \omega]$.

- Suppose $\{b_n : n \in \omega\}$ is a family of pairwise disjoint partitioners for a weakly tight $\mathscr{A}$.
- There cannot be $a \in \mathscr{A}$ which has infinite intersection with b_n and b_m for distinct n and m.
- However Shelah's method is explicitly designed to produce many pairwise disjoint partitioners (picture on board).

- Suppose $\{b_n : n \in \omega\}$ is a family of pairwise disjoint partitioners for a weakly tight $\mathscr{A}$.
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- However Shelah's method is explicitly designed to produce many pairwise disjoint partitioners (picture on board).
- Solution: make two changes to the basic framework.
- First, each member of the a.d. family will be associated with a countable collection of nodes, and will be the union of a countable sequence of infinite subsets of ω.
- Second, each such countable sequence will be associated with its own node, and the collection $\mathcal{I}_\eta$ of countable sequences allowable at a node η will be chosen carefully.

Theorem (R. and Steprans[1])

Assume $\mathfrak{s} \leq \mathfrak{b}$. Then there is a weakly tight family.

- As always fix an (ω, ω)-splitting family $\langle e_\alpha : \alpha < \mathfrak{s} \rangle$.

Definition

We say that a sequence $\vec{C} = \langle c_n : n \in \omega \rangle \subset [\omega]^\omega$ is a p.w.d. if for any $n \neq m$, $c_n \cap c_m = 0$. $\vec{C}(n)$ denotes c_n. For an $\eta \in 2^{\leq \mathfrak{s}}$, we define

$$\mathcal{I}_\eta = \left\{ \vec{C} : \vec{C} \text{ is p.w.d. and } \forall \beta < \operatorname{dom}(\eta) \forall^\infty n \in \omega \left[\vec{C}(n) \subset e_\beta^{\eta(\beta)} \right] \right\}.$$

- At a stage $\alpha < \mathfrak{c}$, we are given an increasing sequence $\langle \mathcal{T}_\beta : \beta < \alpha \rangle$ of subtrees of $2^{<\kappa}$, as well as an almost disjoint family $\mathscr{A}_\alpha = \{a_\beta : \beta < \alpha\}$.
- We ensure that for each $\beta < \alpha$, $a_\beta = \bigcup_{n \in \omega} d_n^\beta$, where $\vec{D}^\beta = \langle d_n^\beta : n \in \omega \rangle$ is a p.w.d.

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- We ensure that for each $\beta < \alpha$, $a_\beta = \bigcup_{n \in \omega} d^\beta_n$, where $\vec{D}^\beta = \langle d^\beta_n : n \in \omega \rangle$ is a p.w.d.
- Moreover, to each a_β and each d^β_n, we associate nodes $\eta(a_\beta) \in \mathcal{T}_\beta$ and $\eta(d^\beta_n) \in \mathcal{T}_\beta$ such that the following conditions hold:
  1. $\vec{D}^\beta \in \mathcal{I}_{\eta(a_\beta)}$;
  2. $\forall \gamma < \mathrm{dom}\,(\eta(d^\beta_n)) \left[d^\beta_n \subset^* e^{\eta(d^\beta_n)(\gamma)}_\gamma \right]$.

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- Moreover, to each a_β and each d_n^β, we associate nodes $\eta(a_\beta) \in \mathcal{T}_\beta$ and $\eta(d_n^\beta) \in \mathcal{T}_\beta$ such that the following conditions hold:
    1. $\vec{D}^\beta \in \mathcal{I}_{\eta(a_\beta)}$;
    2. $\forall \gamma < \mathrm{dom}\,(\eta(d_n^\beta)) \left[d_n^\beta \subset^* e_\gamma^{\eta(d_n^\beta)(\gamma)} \right]$.
- Important that $\eta(a_\beta) \neq \eta(a_\gamma)$ for all $\gamma < \beta < \alpha$.
- Also $\eta(d_n^\beta) \neq \eta(d_m^\gamma)$ for all $\langle \beta, n \rangle \neq \langle \gamma, m \rangle$ where $\beta, \gamma < \alpha$, and $n, m \in \omega$,
- Finally $\eta(a_\beta) \neq \eta(d_m^\gamma)$ for all $\beta, \gamma < \alpha$, and $m \in \omega$.

- $\bigcup_{\beta<\alpha} \mathcal{T}_\beta = \left\{\sigma \in 2^{<\kappa} : \exists \xi < \alpha \left[\sigma \subset \eta(a_\xi) \vee \exists n \in \omega \left[\sigma \subset \eta(d_n^\xi)\right]\right]\right\}$.
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- Thus $\bigcup_{\beta<\alpha} \mathcal{T}_\beta$ is the union of $< \mathfrak{c}$ chains.

Lemma

Let $b \in \mathcal{I}^+(\mathscr{A}_\delta)$. There is a $c \in [b]^\omega$ which is a.d. from a_β for every $\beta < \alpha$, and a $\tau \in (2^{<\mathfrak{s}}) \setminus \left(\bigcup_{\beta<\alpha} \mathcal{T}_\beta\right)$ such that $\forall \delta < \operatorname{dom}(\tau) \left[c \subset^* e_\delta^{\tau(\delta)}\right]$.

- The proof is just as before, but just a slight twist.
- Before we relied on the fact that if the node associated with a_α and the node associated with a_β are incomparable, then a_α and a_β are automatically a.d
- This is not true anymore.

- But because of the way we have set things up, it is enough to have $\tau \notin \left(\bigcup_{\beta<\alpha}\mathcal{T}_\beta\right)$.
- By the usual construction we can arrange to have $\tau \notin \left(\bigcup_{\beta<\alpha}\mathcal{T}_\beta\right)$, as well as $c \in [b]^\omega$ such that $\forall \xi < \mathrm{dom}(\tau)\left[c \subset^* e^{\tau(\xi)}_{\tau\restriction\xi}\right]$ *and* $\left|c \cap d^\beta_n\right| < \omega$ for all $\beta < \alpha$ and $n \in \omega$.

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- Now, if τ and η_β are incomparable, then for some $\xi < \operatorname{dom}(\tau)$, $c \subset^* e^{\tau(\xi)}_{\tau \restriction \xi}$ and there is some $n \in \omega$ such that $\forall m \geq n \left[d^\beta_m \subset e^{1-\tau(\xi)}_{\tau \restriction \xi}\right]$. So $\left|c \cap \left(\bigcup_{m \geq n} d^\beta_m\right)\right| < \omega$.
- On the other hand, for any $m < n$, $\left|c \cap d^\beta_m\right| < \omega$. So $\left|c \cap \left(\bigcup_{m<n} d^\beta_m\right)\right| < \omega$.

- Now we can prove the theorem.
- We are at a stage $\alpha < \mathfrak{c}$ and we are given $\{b_n : n \in \omega\} \subset [\omega]^\omega$ such that for each $n \in \omega$, $b_n \in \mathcal{I}^+(\mathscr{A}_\alpha)$.
- Applying a previous lemma find $c_n \in [b_n]^\omega$ and nodes $\tau_n \in 2^{<\mathfrak{s}}$ such that
  1. c_n is a.d. from a_β for all $\beta < \alpha$;
  2. $\forall \xi < \mathrm{dom}(\tau_n) \left[c_n \subset^* e_\xi^{\tau_n(\xi)}\right]$;
  3. $\tau_n \notin \bigcup_{\beta<\alpha} \mathcal{T}_\beta$ and $\forall m < n \, [\tau_n \not\subset \tau_m]$.

- WLOG the $\vec{C}_0 = \langle c_n : n \in \omega \rangle$ is a p.w.d.
- Look for least $\gamma_0 < \mathfrak{s}$ such that $\exists^{\infty} n \in \omega \left[\left| c_n \cap e^0_{\gamma_0} \right| = \omega \right]$ and $\exists^{\infty} n \in \omega \left[\left| c_n \cap e^1_{\gamma_0} \right| = \omega \right]$.
- There is a unique $\tau_0 \in 2^{\alpha_0}$ such that

$$\forall \xi < \alpha_0 \forall i \in 2 \left[\tau_0(\xi) = i \leftrightarrow \exists^{\infty} n \in \omega \left[\left| c_n \cap e^i_{\xi} \right| = \omega \right] \right].$$

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- Look for least $\gamma_0 < \mathfrak{s}$ such that $\exists^\infty n \in \omega \left[\left| c_n \cap e^0_{\gamma_0} \right| = \omega \right]$ and $\exists^\infty n \in \omega \left[\left| c_n \cap e^1_{\gamma_0} \right| = \omega \right]$.
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- Proceeding in this way, build sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset \mathfrak{s}$, $\langle \tau_s : s \in 2^{<\omega} \rangle \subset 2^{<\mathfrak{s}}$, $\langle \vec{C}_s : s \in 2^{<\omega} \rangle$, and $\langle z_s : s \in 2^{<\omega} \rangle \subset [\omega]^\omega$ such that:
  1. $\forall s \in 2^{<\omega} \forall i \in 2 \left[\alpha_s = \operatorname{dom}(\tau_s) \wedge \alpha_{s^\frown\langle i \rangle} > \alpha_s \wedge \tau_{s^\frown\langle i \rangle} \supset \tau_s{}^\frown \langle i \rangle \right]$;
  2. The domain of $\vec{C}_s = z_s$ (so $z_0 = \omega$) and $z_{s^\frown\langle i \rangle} \subset z_s$;
  3. For all $n \in z_{s^\frown\langle i \rangle} \left[\vec{C}_{s^\frown\langle i \rangle}(n) \subset \vec{C}_s(n) \right]$
  4. for each $s \in 2^{<\omega}$ and for each $\xi < \alpha_s$, $\forall^\infty n \in z_s \left[e^{1-\tau_s(\xi)}_\xi \cap \vec{C}_s(n) \right| < \omega$;

(5) for each $s \in 2^{<\omega}$, both $\exists^{\infty} n \in \omega \left[\left| e^{0}_{\alpha_s} \cap \vec{C}_s(n) \right| = \omega \right]$ and $\exists^{\infty} n \in \omega \left[\left| e^{1}_{\alpha_s} \cap \vec{C}_s(n) \right| = \omega \right]$;

(6) for $n \in z_{s^\frown\langle i \rangle}$, $\vec{C}_{s^\frown\langle i \rangle}(n) = \vec{C}_s(n) \cap e^{i}_{\alpha_s}$.

- For each $f \in 2^\omega$, put $\alpha_f = \sup \left\{\alpha_{(f \restriction n)} : n \in \omega\right\}$ and $\tau_f = \bigcup_{n \in \omega} \tau_{(f \restriction n)}$.
- Again, we have $\alpha_f < \kappa$.

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- Again, we have $\alpha_f < \kappa$.
- Again we can find $f \in 2^{\omega}$ such that $\tau_f \notin \bigcup_{\beta < \alpha} \mathcal{T}_{\beta} \cup \{\sigma : \exists n \in \omega\, [\sigma \subset \tau_n]\}$.
- Take a $z = k_0 < k_1 < \cdots$ such that $\forall n \in \omega \left[k_n \in z_{f \restriction n}\right]$. For each $n \in \omega$ define $\vec{E}(k_n) = \vec{C}_{f \restriction (n)}(k_n)$.

- for each $\delta < \alpha_f$, define a function $f_\delta : z \to \omega$ by

$$f_\delta(n) = \begin{cases} \max\left(\vec{E}(n) \cap e_\delta^{1-\tau_f(\delta)}\right) & \text{if } \left|\vec{E}(n) \cap e_\delta^{1-\tau_f(\delta)}\right| < \omega \\ 0 & \text{otherwise} \end{cases}$$

The second case only occurs finitely often.

- Also let G be the set of $\beta < \alpha$ such that either $\eta(a_\beta) \subsetneq \tau_f$ or that there is an $m \in \omega$ so that $\eta(d_m^\beta) \subsetneq \tau_f$.
- $|G| \leq \left|\alpha_f\right| < \mathfrak{s} \leq \mathfrak{b}$.
- a_β is a.d. from $\vec{E}(k_n)$ for each $n \in \omega$ and each $\beta \in G$. So each $\beta \in G$ determines a function $g_\beta : z \to \omega$

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- a_β is a.d. from $\vec{E}(k_n)$ for each $n \in \omega$ and each $\beta \in G$. So each $\beta \in G$ determines a function $g_\beta : z \to \omega$
- $\{f_\delta : \delta < \alpha_f\}$ is a collection of functions of size at most $< \mathfrak{s} \leq \mathfrak{b}$.
- Find $f \in \omega^z$ such that $\forall \delta < \alpha_f \, [f_\delta <^* f]$.
- For each $n \in \omega$ define $D^\alpha(n) = \vec{E}(k_n) \setminus f(k_n)$.
- $a_\alpha = \bigcup_{n \in \omega} D^\alpha(n)$ and $\eta(a_\alpha) = \tau_f$.

Bibliography

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