Constructing special almost disjoint families

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Outline

1. A completely separable family from $s \leq \alpha$

2. A completely separable family from $c < \aleph_\omega$
Building a completely separable family

**Theorem (Mildenberger, R., and Steprans [1])**

If $s \leq \alpha$, then there is a completely separable family.

- The basic framework is contained in this proof. It is also the simplest.
- Easy to see that a completely separable family exists if $\alpha = c$.
- (Balcar, Simon, Vojtas): They exist if any one of these holds: $s = \omega_1$, $b = d$, or $d \leq \alpha$.
- The hypothesis $s \leq \alpha$ is weaker than all of the above.
A completely separable family from $s \leq \alpha$  
A completely separable family from $c < \aleph_\omega$

Bibliography

Building a completely separable family

- $\mathcal{F} \subset \mathcal{P}(\omega)$ is said to be $(\omega, \omega)$-splitting if for each collection 
  \[ \{b_n : n \in \omega\} \subset [\omega]^\omega, \text{ there exists } a \in \mathcal{F} \text{ such that} \]
  \[ \exists^\infty n \in \omega \ [|a \cap b_n| = \omega] \text{ and } \exists^\infty n \in \omega \ [|(\omega \setminus a) \cap b_n| = \omega]. \]

Definition

\[ s_{\omega,\omega} = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{P}(\omega) \land \mathcal{F} \text{ is } (\omega, \omega) - \text{splitting}\}. \]

- Note that $s \leq s_{\omega,\omega}$ is clear.
Building a completely separable family

Lemma

\[ s = s_{\omega, \omega}. \]

Proof.

Case 1: \( s < b \). Let \( \langle e_\alpha : \alpha < \kappa \rangle \) be a splitting family. Suppose it is not \((\omega, \omega)\)-splitting. Fix \( \{ b_n : n \in \omega \} \) witnessing this. In other words, for each \( \alpha < \kappa \) there is \( i_\alpha \in 2 \) such that \( \forall n \in \omega \left[ |b_n \cap e_{\alpha i_\alpha}| < \omega \right] \). WLOG, the \( b_n \) are pairwise disjoint. Now, for each \( \alpha < s \) define \( f_\alpha \in \omega^\omega \) as follows:

\[
f_\alpha(n) = \begin{cases} 
\sup(b_n \cap e_{\alpha i_\alpha}) & \text{if } |b_n \cap e_{\alpha i_\alpha}| < \omega \\
0 & \text{otherwise}
\end{cases}
\]
Building a completely separable family

Proof.

By hypothesis the first case occurs for all but finitely many $n$. Since $s < b$, find $f \in \omega^\omega$ such that $\forall \alpha < s [f_\alpha \leq^* f]$. Choose $k_n \in b_n$ such that $k_n > f(n)$. Then $\{k_n : n \in \omega\}$ is an infinite set not split by any $e_\alpha$. 
Building a completely separable family

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Case 2: $b \leq s$. Proof by picture on the board.
Building a completely separable family

Proof.

By hypothesis the first case occurs for all but finitely many $n$. Since $s < b$, find $f \in \omega^\omega$ such that $\forall \alpha < s [f_\alpha \leq^* f]$. Choose $k_n \in b_n$ such that $k_n > f(n)$. Then \{ $k_n : n \in \omega$ \} is an infinite set not split by any $e_\alpha$.

Case 2: $b \leq s$. Proof by picture on the board.

Lemma

If $\langle e_\alpha : \alpha < s \rangle$ is $(\omega, \omega)$-splitting, then for any infinite a.d. family $\mathcal{A} \subset [\omega]^\omega$ and for any $b \in I^+(\mathcal{A})$, there is $\alpha < s$ such that $b \cap e_\alpha^0 \in I^+(\mathcal{A})$ and $b \cap e_\alpha^1 \in I^+(\mathcal{A})$. 
Building a completely separable family

Lemma

Let \( \langle e_\alpha : \alpha < \kappa \rangle \) witness \( \kappa = s_{\omega, \omega} \). Let \( \mathcal{A} \subset [\omega]^\omega \) be any a.d. family. Then for each \( b \in I^+(\mathcal{A}) \), there is an \( \alpha < \kappa \) such that \( b \cap e_\alpha^0 \in I^+(\mathcal{A}) \) and \( b \cap e_\alpha^1 \in I^+(\mathcal{A}) \).

Proof.

We may assume that there exist an infinite set \( \{a_n : n \in \omega\} \subset \mathcal{A} \) such that \( \forall n \in \omega \quad |a_n \cap b| = \omega \) (otherwise it is easy). Let \( \alpha < \kappa \) be such that \( \exists^\infty n \in \omega \quad |e_\alpha^0 \cap a_n \cap b| = \omega \) and \( \exists^\infty n \in \omega \quad |e_\alpha^1 \cap a_n \cap b| = \omega \). \( \alpha \) is as needed.
Building a completely separable family

- Say $\kappa = s = s_{\omega,\omega}$ and say $\langle x_\alpha : \alpha < \kappa \rangle$ is an $(\omega, \omega)$-splitting family.
Building a completely separable family

Say $\kappa = s = s_{\omega,\omega}$ and say $\langle x_\alpha : \alpha < \kappa \rangle$ is an $(\omega, \omega)$-splitting family.

Construct $\langle a_\alpha : \alpha < c \rangle$ and $\langle \sigma_\alpha : \alpha < c \rangle \subset 2^{<\kappa}$ such that:

1. $\forall \alpha < c \forall \xi < \text{dom}(\sigma_\alpha) \left[ a_\alpha \subseteq \ast_{\xi} x_{\sigma_\alpha}(\xi) \right]$;
2. $\forall \alpha < \beta < c \left[ \sigma_\alpha \neq \sigma_\beta \right]$.

Observe that if $\alpha \neq \beta$, then by (2), $a_\alpha$ and $a_\beta$ are a.d. unless $\sigma_\alpha$ and $\sigma_\beta$ are comparable.
Building a completely separable family

Say $\kappa = s = s_{\omega, \omega}$ and say $\langle x_\alpha : \alpha < \kappa \rangle$ is an $(\omega, \omega)$-splitting family.

Construct $\langle a_\alpha : \alpha < c \rangle$ and $\langle \sigma_\alpha : \alpha < c \rangle \subset 2^{<\kappa}$ such that:

1. $\forall \alpha < c \forall \xi < \text{dom}(\sigma_\alpha) \left[ a_\alpha \subset^* x_\xi^{\sigma_\alpha(\xi)} \right]$;
2. $\forall \alpha < \beta < c \left[ \sigma_\alpha \neq \sigma_\beta \right]$.

Observe that if $\alpha \neq \beta$, then by (2), $a_\alpha$ and $a_\beta$ are a.d. unless $\sigma_\alpha$ and $\sigma_\beta$ are comparable.

Main point: At a stage $\delta < c$ $\mathcal{A}_\delta = \{ a_\alpha : \alpha < \delta \}$ is “nowhere MAD” – i.e. if $b \in \mathcal{I}^+(\{a_\alpha : \alpha < \delta\})$, then there is $a \in [b]^{\omega}$ such that $\forall \alpha < \delta \left[ |a \cap a_\alpha| < \omega \right]$ (and also a node $\sigma$ associated with $a$).
Building a completely separable family

- If $b \in I^+(\mathcal{A}_\delta)$, then look for least $\alpha_0 < \kappa$ such that $b \cap x^0_{\alpha_0} \in I^+(\mathcal{A}_\delta)$ and $b \cap x^1_{\alpha_0} \in I^+(\mathcal{A}_\delta)$.
- There is a unique $\tau_0 \in 2^{\alpha_0}$ such that
  \[
  \forall \xi < \alpha_0 \forall i \in 2 \left[ \tau_0(\xi) = i \iff b \cap x^i_\xi \in I^+(\mathcal{A}_\delta) \right].
  \]
Building a completely separable family

- If $b \in \mathcal{I}^+(\mathcal{A}_\delta)$, then look for least $\alpha_0 < \kappa$ such that $b \cap x^0_{\alpha_0} \in \mathcal{I}^+(\mathcal{A}_\delta)$ and $b \cap x^1_{\alpha_0} \in \mathcal{I}^+(\mathcal{A}_\delta)$.

- There is a unique $\tau_0 \in 2^{\alpha_0}$ such that

$$\forall \xi < \alpha_0 \forall i \in 2 \left[ \tau_0(\xi) = i \leftrightarrow b \cap x^i_\xi \in \mathcal{I}^+(\mathcal{A}_\delta) \right].$$

- Proceeding in the same way one can build two sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset \kappa$ and $\langle \tau_s : s \in 2^{<\omega} \rangle \subset 2^{<\kappa}$ such that:

1. $\forall s \in 2^{<\omega} \forall i \in 2 \left[ \alpha_s = \text{dom}(\tau_s) \land \alpha_s-\langle i \rangle > \alpha_s \land \tau_s-\langle i \rangle \supset \tau_s-\langle i \rangle \right]$;
2. for each $s \in 2^{<\omega}$ and for each $\xi < \alpha_s$, $x^1_{\xi} - \tau_s(\xi) \cap b \cap \left( \bigcap_{t \prec s} x^s_{\tau_s(\alpha_t)} \right) \in \mathcal{I}(\mathcal{A}_\delta)$;
3. for each $s \in 2^{<\omega}$, both $x^0_{\alpha_s} \cap b \cap \left( \bigcap_{t \prec s} x^s_{\tau_s(\alpha_t)} \right) \in \mathcal{I}^+(\mathcal{A}_\delta)$ and $x^1_{\alpha_s} \cap b \cap \left( \bigcap_{t \prec s} x^s_{\tau_s(\alpha_t)} \right) \in \mathcal{I}^+(\mathcal{A}_\delta)$. 

Dilip Raghavan

Constructing special almost disjoint families
Building a completely separable family

- For each \( f \in 2^\omega \), put \( \alpha_f = \sup \{ \alpha(f \upharpoonright n) : n \in \omega \} \) and \( \tau_f = \bigcup_{n \in \omega} \tau(f \upharpoonright n) \).
- Note \( \alpha_f < \kappa \).
Building a completely separable family

- For each $f \in 2^\omega$, put $\alpha_f = \sup \{ \alpha(f \restriction n) : n \in \omega \}$ and $\tau_f = \bigcup_{n \in \omega} \tau(f \restriction n)$.
- Note $\alpha_f < \kappa$.
- Find $f \in 2^\omega$ such that $\tau_f \notin \{ \sigma \in 2^{<\kappa} : \exists \alpha < \delta [\sigma \subset \sigma_\alpha] \}$.
- $e \in [b]^\omega \cap I^+(\mathcal{A}_\delta)$ such that $\forall n \in \omega \[e \subset^* e_n\]$, where $e_n = b \cap \left( \bigcap_{m < n} \tau_f(\alpha(f \restriction m)) \right)$. 

Dilip Raghavan

Constructing special almost disjoint families
Building a completely separable family

- For any $\xi < \alpha_f$, there is $F_{\xi} \in [\delta]^{<\omega}$ such that
  \[
  \left( x_{\xi}^{1-\tau_f(\xi)} \cap e \right) \subset^* \left( \bigcup_{\alpha \in F_{\xi}} a_\alpha \right).
  \]

- Consider $\mathcal{F} = \bigcup_{\xi < \alpha_f} F_{\xi}$ and $\mathcal{G} = \{ \alpha < \delta : \sigma_\alpha \subset \tau_f \}$.

- $|\mathcal{F} \cup \mathcal{G}| < \kappa \leq \alpha$. 

Dilip Raghavan
Constructing special almost disjoint families
Building a completely separable family

- For any $\xi < \alpha_f$, there is $F_\xi \in [\delta]^{<\omega}$ such that
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- Consider $\mathcal{F} = \bigcup_{\xi < \alpha_f} F_\xi$ and $\mathcal{G} = \{\alpha < \delta : \sigma_\alpha \subset \tau_f\}$.
- $|\mathcal{F} \cup \mathcal{G}| < \kappa \leq \alpha$.
- So there is $a \in [e]^{\omega}$ such that $\forall \alpha \in \mathcal{F} \cup \mathcal{G} \ [|a \cap a_\alpha| < \omega]$. 
Building a completely separable family

For any $\xi < \alpha_f$, there is $F_\xi \in [\delta]^{<\omega}$ such that

$$\left( x_\xi^{1-\tau_f(\xi)} \cap e \right) \subset^* \left( \bigcup_{\alpha \in F_\xi} a_\alpha \right).$$

Consider $F = \bigcup_{\xi < \alpha_f} F_\xi$ and $G = \{\alpha < \delta : \sigma_\alpha \subset \tau_f\}$.

$|F \cup G| < \kappa \leq \alpha$.

So there is $a \in [e]^{\omega}$ such that $\forall \alpha \in F \cup G [\|a \cap a_\alpha\| < \omega]$.

Now $a$ and $\sigma_f$ are as needed:

1. If $\alpha \in G$, then $a$ and $a_\alpha$ are a.d. by choice.
2. If $\alpha \notin G$, then $a_\alpha$ and $a$ are a.d. because $\forall \xi < \alpha_f [a \subset^* x_\xi^{\sigma_f(\xi)}]$. 
The case \( \alpha < \mathfrak{s} \)

- When \( \alpha \) is small, \( \mathfrak{b} \) is also small.
- Key point: there is a small collection of sets that splits any set of a specific form (even though there are no small splitting families).
The case $\alpha < \varsigma$

- When $\alpha$ is small, $b$ is also small.
- Key point: there is a small collection of sets that splits any set of a specific form (even though there are no small splitting families).

Lemma

Let $\langle c_n : n \in \omega \rangle$ be pairwise disjoint elements of $[\omega]^{\omega}$. Then there is a collection $\langle x_\alpha : \alpha < b \rangle \subseteq P(\omega)$ such that for any $b \in [\omega]^\omega$ and any infinite a.d. family $\mathcal{A} \subseteq [\omega]^{\omega}$, if for all $n \in \omega$ and for all $f \in \omega^\omega$, $\bigcup_{m \geq n} \{ k \in b \cap c_m : k > f(m) \} \in I^+(\mathcal{A})$, then there is $\alpha < b$ such that $x_\alpha^0 \cap b \in I^+(\mathcal{A})$ and $x_\alpha^1 \cap b \in I^+(\mathcal{A})$. 

Dilip Raghavan

Constructing special almost disjoint families
The case $\alpha < \mathfrak{s}$

Proof.

Fix a $\prec^*$-increasing everywhere unbounded family $\langle f_\alpha : \alpha < b \rangle \subset \omega^\omega$. For each $\alpha < b$ and $n \in \omega$, let $x_{\alpha,n} = \{k \in c_n : k \leq f_\alpha(n)\}$. Let $x_\alpha = \bigcup_{n \in \omega} x_{\alpha,n}$. Why does this work?
The case $\alpha < \varsigma$

**Proof.**

Fix a $<^*$-increasing everywhere unbounded family $\langle f_\alpha : \alpha < \varsigma \rangle \subset \omega^\omega$. For each $\alpha < \varsigma$ and $n \in \omega$, let $x_{\alpha,n} = \{k \in c_n : k \leq f_\alpha(n)\}$. Let $x_\alpha = \bigcup_{n \in \omega} x_{\alpha,n}$. Why does this work? Take any $b \in [\omega]^\omega$ and any infinite a.d. family $\mathcal{A} \subset [\omega]^\omega$. Assume that $b$ satisfies the hypothesis. In particular, for each $n \in \omega$, $\bigcup_{m \geq n} (b \cap c_m)$ is $\mathcal{I}(\mathcal{A})$-positive. So we can find $d \in \big[\bigcup_{n \in \omega} (b \cap c_n)\big]^\omega \cap \mathcal{I}^+(\mathcal{A})$ such that $\forall n \in \omega \ [|d \cap c_n| < \omega]$. Now there are formally 2 cases:
The case $\alpha < \mathfrak{s}$

Proof.

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Case I: there is $e \in [d]^\omega$ which is a.d. from every $a \in \mathcal{A}$. Let $X = \{m \in \omega : e \cap c_m \neq 0\}$. Define $f : X \to \omega$ by $f(m) = \min(e \cap c_m)$. There is $\alpha < b$ such that $\exists^\infty m \in X \ [f(m) \leq f_\alpha(m)]$. For any such $m \in X$, $x_{\alpha,m} \cap e \neq 0$. So $|x_\alpha^0 \cap e| = \omega$. This implies $x_\alpha^0 \cap d$, and hence $x_\alpha^0 \cap b$ are in $\mathcal{I}^+(\mathcal{A})$. On the other hand, $x_\alpha^1 \cap b \in \mathcal{I}^+(\mathcal{A})$ by hypothesis.

$\blacksquare$
The case $a < \mathfrak{s}$

Proof.

Case II: there are infinitely many $a \in \mathcal{A}$ such that $|a \cap d| = \omega$. Fix such a family $\{a_n : n \in \omega\} \subset \mathcal{A}$. For each $n \in \omega$, let $X_n = \{m \in \omega : a_n \cap d \cap c_m \neq 0\}$. There is $\alpha < b$ such that for each $n \in \omega$,

$$\exists^\infty m \in X_n [c_m \cap d \cap a_n \cap (f_\alpha(m) + 1) \neq 0].$$

Then for each $n \in \omega$,

$$|a_n \cap d \cap x^0_\alpha| = \omega.$$  So $d \cap x^0_\alpha$ and hence $b \cap x^0_\alpha$ are in $\mathcal{I}^+(\mathcal{A})$. $x^1_\alpha \cap b$ is in $\mathcal{I}^+(\mathcal{A})$ by hypothesis.

$\dashv$
The case $\alpha < \varsigma$

- In a sense we only care about splitting things that hit infinitely many $c_n$, for some collection $\langle c_n : n \in \omega \rangle$. 
The case $\alpha < \delta$

- In a sense we only care about splitting things that hit infinitely many $c_n$, for some collection $\langle c_n : n \in \omega \rangle$.
- There is a problem: the collection $\langle c_n : n \in \omega \rangle$ that we care about will keep changing at every stage of the construction.
- Solution: make the tree more complicated.
The case $\alpha < \varsigma$

- In a sense we only care about splitting things that hit infinitely many $c_n$, for some collection $\langle c_n : n \in \omega \rangle$.
- There is a problem: the collection $\langle c_n : n \in \omega \rangle$ that we care about will keep changing at every stage of the construction.
- Solution: make the tree more complicated.
- Main difference: instead of using a sequence of sets $\langle e_\alpha : \alpha < \kappa \rangle$, use a tree of sets $\langle e_\eta : \eta \in 2^{<\kappa} \rangle$.
- The pair $e^0, e^1$ used at a node of the tree now depends not just on the height of that node, but also on all the pairs of sets that occur below that node.
The case $\alpha < \mathfrak{s}$

- Along each (long enough) branch $\psi$ of the tree, each countable subset of $\psi$ can be “captured” at some node $\eta$ that lies on $\psi$.
- This “captured” countable set determines a collection $\langle c_n : n \in \omega \rangle$.
- The sets that hit infinitely many of the $c_n$ will be split using a small family before $\psi$ is reached.
The case $\alpha < \varsigma$

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- This “captured” countable set determines a collection $\langle c_n : n \in \omega \rangle$.
- The sets that hit infinitely many of the $c_n$ will be split using a small family before $\psi$ is reached.
- The assumption that $\varsigma < \aleph_\omega$ becomes relevant for capturing the countable sets.
The case $\alpha < \mathfrak{s}$

**Definition**

Let $\kappa$ be any cardinal. A set $X \subset [\kappa]^{\leq \omega}$ is called cofinal if

$\forall a \in [\kappa]^{\leq \omega} \exists b \in X \ [a \subset b]$.

$$
\text{cf}(\langle [\kappa]^{\leq \omega}, \subset \rangle) = \min \{|X| : X \subset [\kappa]^{\leq \omega} \text{ is cofinal}\}.
$$
The case $\alpha < \varsigma$

**Definition**

Let $\kappa$ be any cardinal. A set $X \subset [\kappa]^{\leq \omega}$ is called cofinal if

$$\forall a \in [\kappa]^{\leq \omega} \exists b \in X [a \subset b].$$

$$\text{cf}(\langle [\kappa]^{\leq \omega}, \subset \rangle) = \min \left\{ |X| : X \subset [\kappa]^{\leq \omega} \text{ is cofinal} \right\}.$$

- For any $n < \omega$, $\text{cf}(\langle [\aleph_n]^{\leq \omega}, \subset \rangle) = \aleph_n$ (obvious for $\aleph_1$; by induction for larger $n$).
- So for any $n < \omega$, there is a sequence $\langle u_\alpha : \omega \leq \alpha < \aleph_n \rangle$ such that
  1. $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$;
  2. if $X \subset \aleph_n$ is any uncountable set, then there exists $\alpha < \text{sup}(X)$ such that $|u_\alpha \cap X| = \omega$.
- If in addition you know that $b \leq \aleph_n$, then you can strengthen 1 to say that $\text{otp}(u_\alpha) = \omega$; but 2 will only apply to sets order type at least $b$. 

Dilip Raghavan  
Constructing special almost disjoint families
The case $\alpha < \varsigma$

Definition

For cardinals $\kappa > \lambda > \omega$, $P(\kappa, \lambda)$ says that there is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ such that

1. $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$
2. For each $X \subset \kappa$, if $X$ is bounded in $\kappa$ and $\text{otp}(X) = \lambda$, then $\exists \omega \leq \alpha < \text{sup}(X) [u_\alpha \cap X| = \omega]$. 
The case $\alpha < s$

**Definition**

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$\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ such that

1. $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$

2. for each $X \subset \kappa$, if $X$ is bounded in $\kappa$ and $\text{otp}(X) = \lambda$, then

$$\exists \omega \leq \alpha < \sup(X) \ [|u_\alpha \cap X| = \omega].$$

- If $b \leq \lambda < \kappa < \beth_\omega$, then $P(\kappa, \lambda)$ is true.

**Theorem (Shelah, 2010 [2])**

If $\alpha < s$ and $P(s, \alpha)$ holds, then there is a completely separable family.

- Forcing the failure of the hypothesis needs large cardinals (and unknown if $\alpha > \omega_1$).
The case $\alpha < \mathfrak{s}$

At a stage $\delta < \mathfrak{c}$ we have $\mathcal{A}_\delta = \langle a_\alpha : \alpha < \delta \rangle$, a subtree $\mathcal{T}_\delta \subset 2^{< \mathfrak{s}}$, a labeling $\langle e_\eta : \eta \in \mathcal{T}_\delta \rangle$, and a sequence of nodes $\langle \eta_\alpha : \alpha < \delta \rangle \subset \mathcal{T}_\delta$ such that for each $\alpha < \delta$:

1. $\forall \xi \in \text{dom}(\eta_\alpha) \left[ a_\alpha \subset^* e_\eta^{\eta_\alpha(\xi)} \right]$;
The case $\alpha < s$

At a stage $\delta < c$ we have $A_\delta = \langle a_\alpha : \alpha < \delta \rangle$, a subtree $T_\delta \subset 2^{<s}$, a labeling $\langle e_\eta : \eta \in T_\delta \rangle$, and a sequence of nodes $\langle \eta_\alpha : \alpha < \delta \rangle \subset T_\delta$ such that for each $\alpha < \delta$:

1. $\forall \xi \in \text{dom}(\eta_\alpha) \left[ a_\alpha \subseteq^* e_\eta^{\eta_\alpha(\xi)} \right]$;
2. If $\sigma \in 2^s$ and if $\sigma \upharpoonright \xi \in T_\delta$ for all $\xi < s$, then $\{ e_{\sigma \upharpoonright \xi} : \xi < s \}$ is an $(\omega, \omega)$-splitting family;
The case $\alpha < s$

- At a stage $\delta < c$ we have $A_\delta = \langle a_\alpha : \alpha < \delta \rangle$, a subtree $T_\delta \subset 2^{<s}$, a labeling $\langle e_\eta : \eta \in T_\delta \rangle$, and a sequence of nodes $\langle \eta_\alpha : \alpha < \delta \rangle \subset T_\delta$ such that for each $\alpha < \delta$:
  1. $\forall \xi \in \text{dom}(\eta_\alpha) \left[ a_\alpha \subseteq^* e_{\eta_\alpha}(\xi) \right]$;
  2. if $\sigma \in 2^s$ and if $\sigma \upharpoonright \xi \in T_\delta$ for all $\xi < s$, then $\{e_{\sigma \upharpoonright \xi} : \xi < s\}$ is an $(\omega, \omega)$-splitting family;
  3. $|T_\delta| < c$ (more precisely $T_\delta$ is the union of $< c$ chains) and $e_{\eta_\alpha} = a_\alpha$;
At a stage $\delta < \kappa$ we have $\mathcal{A}_\delta = \langle a_\alpha : \alpha < \delta \rangle$, a subtree $T_\delta \subset 2^{<\tau}$, a labeling $\langle e_\eta : \eta \in T_\delta \rangle$, and a sequence of nodes $\langle \eta_\alpha : \alpha < \delta \rangle \subset T_\delta$ such that for each $\alpha < \delta$:

1. $\forall \xi \in \text{dom}(\eta_\alpha) \left[ a_\alpha \subseteq^* e_{\eta_\alpha}^\xi \right]$;
2. if $\sigma \in 2^\tau$ and if $\sigma \upharpoonright \xi \in T_\delta$ for all $\xi < \tau$, then $\{ e_{\sigma \upharpoonright \xi} : \xi < \tau \}$ is an $(\omega, \omega)$-splitting family;
3. $|T_\delta| < \kappa$ (more precisely $T_\delta$ is the union of $< \kappa$ chains) and $e_{\eta_\alpha} = a_\alpha$;
4. For $\xi < \tau$, $\eta \in 2^{\xi} \cap T_\delta$, a set $a \subset \xi$ of order type $\omega$, and $n \in \omega$, we use the notation $c_{\eta,a,n} = \left( \bigcap_{m<n} e_{\eta\upharpoonright a(m)}^{\eta(a(m))} \right) \cap e_{\eta\upharpoonright a(n)}^{1-\eta(a(n))}$, where $a(m)$ denotes the $m$th element of $a$;
5. If $\xi < \tau$ and if $X \subset \xi$ has order type $\alpha$ and $\text{sup}(X) = \xi$, then for any $\eta \in 2^{\xi} \cap T_\delta$, there is $a \subset \xi$ with $\text{otp}(a) = \omega$ such that $|a \cap X| = \omega$ and for every $b \in [\omega]^\omega$ and any infinite a.d. family $\mathcal{A} \subset [\omega]^\omega$, if for all $n \in \omega$ and for all $f \in \omega^\omega$, $\bigcup_{m \geq n} \{ k \in b \cap c_m : k > f(m) \} \in I^+(\mathcal{A})$, then there is $\zeta < \xi$ such that $b \cap e_{\eta\upharpoonright \zeta}^0 \in I^+(\mathcal{A})$ and $b \cap e_{\eta\upharpoonright \zeta}^1 \in I^+(\mathcal{A})$. 

Dilip Raghavan

Constructing special almost disjoint families
The case $\alpha < s$

- Given $\langle u_\alpha : \alpha < s \rangle$ witnessing $P(s, \alpha)$, a family $\langle f_\alpha : \alpha < b \rangle$ witnessing $b < s$, and an $(\omega, \omega)$-splitting family $\langle x_\alpha : \alpha < s \rangle$, arranging (1)-(5) is just a matter of bookkeeping.

- Details of the bookkeeping are not deep (just messy).
The case $\alpha < s$

- Given $\langle u_\alpha : \alpha < s \rangle$ witnessing $P(s, \alpha)$, a family $\langle f_\alpha : \alpha < b \rangle$ witnessing $b < s$, and an $(\omega, \omega)$-splitting family $\langle x_\alpha : \alpha < s \rangle$, arranging (1)-(5) is just a matter of bookkeeping.
- Details of the bookkeeping are not deep (just messy).
- The idea is that along a branch $\eta$, any subset $X$ of order type $\alpha$ will be “trapped” by some $u_\alpha$.
- This $u_\alpha$ determines a collection $\{ c_{\eta, u_\alpha, n} : n \in \omega \}$.
- Together with $\langle f_\beta : \beta < b \rangle$ this gives a family $\{ y_\beta : \beta < b \}$ such that any $b$ that behaves like in the lemma w.r.t. the $c_{\eta, u_\alpha, n}$ is split by one of the $y_\beta$.
- There is enough space to enumerate the $\{ y_\beta : \beta < b \}$ (note: this set does not depend on $X$) along $\eta$; so every $b$ that intersects infinitely many of the $c_{\eta, u_\alpha, n}$ will be split before $\eta$ is reached.
The case $\alpha < \varsigma$

- At a stage $\delta < c$, fix some $b \in I^+(\mathcal{A}_\delta)$
- By clause (2), we can once again build sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset \varsigma$ and $\langle \tau_s : s \in 2^{<\omega} \rangle \subset T_{\delta+1}$ as before.
- As before, for any $f \in 2^\omega$, if $\tau_f = \bigcup_{n \in \omega} \tau(f\upharpoonright n)$ and if $\alpha_f = \sup \{\alpha(f\upharpoonright n) : n \in \omega\}$, then $\alpha_f < \varsigma$, and
  $$b \cap e_{\tau_f(\alpha_{f\upharpoonright 0})} \ni b \cap e_{\tau_f(\alpha_{f\upharpoonright 0})} \cap e_{\tau_f(\alpha_{f\upharpoonright 1})} \ni \cdots$$  is a decreasing sequence of sets in $I^+(\mathcal{A}_\delta)$. 
The case $\alpha < \varsigma$

- At a stage $\delta < c$, fix some $b \in I^+(A_\delta)$
- By clause (2), we can once again build sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset s$ and $\langle \tau_s : s \in 2^{<\omega} \rangle \subset T_{\delta+1}$ as before.
- As before, for any $f \in 2^\omega$, if $\tau_f = \bigcup_{n\in\omega} \tau(f \upharpoonright n)$ and if $\alpha_f = \sup \{ \alpha(f \upharpoonright n) : n \in \omega \}$, then $\alpha_f < \varsigma$, and $b \cap e_{\tau_f(\alpha_f \upharpoonright 0)} \supset b \cap e_{\tau_f(\alpha_f \upharpoonright 1)} \cap e_{\tau_f(\alpha_f \upharpoonright 1)} \supset \cdots$ is a decreasing sequence of sets in $I^+(A_\delta)$.
- Choose $f \in 2^\omega$ such that $\tau_f \notin T_\delta$ and choose $e \in [b]^\omega \cap I^+(A_\delta)$ that is almost included in this decreasing sequence.
The case $\alpha < \varsigma$

- As before, for any $\xi < \alpha_f$, there is a minimal $F_\xi \in [\delta]^{<\omega}$ such that $e \cap e^{1-\tau_f(\xi)} \subseteq^* \bigcup_{\alpha \in F_\xi} a_\alpha$.

- Recall clause (3) which says that for any $\alpha < \delta$, $e_{\eta_\alpha} = a_\alpha$.

- For any $\alpha < \delta$, if $\eta_\alpha \subset \tau_f$, then $\text{dom}(\eta_\alpha) < \alpha_f$ and $\tau_f(\text{dom}(\eta_\alpha)) = 1$ because of this clause.
The case \( \alpha < \varsigma \)

- As before, for any \( \xi < \alpha_f \), there is a minimal \( F_{\xi} \in [\delta]^{<\omega} \) such that 
  \[ e \cap e^{1-\tau_f(\xi)} \subseteq^* \bigcup_{\alpha \in F_{\xi}} a_\alpha. \]
- Recall clause (3) which says that for any \( \alpha < \delta \), \( e_{\eta_\alpha} = a_\alpha \).
- For any \( \alpha < \delta \), if \( \eta_\alpha \subset \tau_f \), then \( \text{dom}(\eta_\alpha) < \alpha_f \) and \( \tau_f(\text{dom}(\eta_\alpha)) = 1 \) because of this clause.
- Conclusion: It is enough to find \( a \in [e]^\omega \) such that 
  \[ \forall \xi < \alpha_f \left[ a \subseteq^* e^{\tau_f(\xi)}(\tau_f) \right]. \]
The case $\alpha < \diamondsuit$

Consider the collection $G$ of all $\zeta < \alpha_f$ for which there is $x \in [e]^{\omega}$ such that:

1. $\forall \xi < \zeta \left[ x \subset^* e^{\tau_f(\xi)}_{(\tau_f) \upharpoonright \xi} \right]$;
2. $x \cap e^{1-\tau_f(\xi)}_{(\tau_f) \upharpoonright \xi}$ is infinite.
The case $\alpha < \varsigma$

Consider the collection $G$ of all $\zeta < \alpha_f$ for which there is $x \in [e]^{\omega}$ such that:

1. $\forall \xi < \zeta \left[ x \subset^* e^{\tau_f(\xi)}_{(\tau_f)\upharpoonright \xi} \right]$;
2. $x \cap e^{1-\tau_f(\zeta)}_{(\tau_f)\upharpoonright \zeta}$ is infinite.

If $|G| < \alpha$, then we can find $a \in [e]^{\omega}$ as needed.

Why? $\left| \bigcup_{\zeta \in G} F_\zeta \right| < \alpha$. Take $a \in [e]^{\omega}$ which is a.d. from everything in $\bigcup_{\zeta \in G} F_\zeta$.

Suppose there exists $\zeta < \alpha_f$ such that $\left| a \cap e^{1-\tau_f(\zeta)}_{(\tau_f)\upharpoonright \zeta} \right| = \omega$. Take the least such $\zeta$. Then $a$ witnesses that $\zeta \in G$, which is a contradiction.
The case $\alpha < \mathfrak{s}$

- So assume that $|G| \geq \alpha$.
- Let $\xi \leq \alpha_f$ be minimal such that $\text{otp}(G \cap \xi) = \alpha$.
- Apply clause (5) to $\xi$ with $\eta = \tau_f \upharpoonright \xi$, $X = G \cap \xi$, to get $a \subset \xi$ of order type $\omega$ with the property given in the clause.
- We wish to use this property of the set $a$ with $b = e$ and $\mathcal{A} = \mathcal{A}_\delta$.
The case $\alpha < \mathfrak{s}$

- So assume that $|G| \geq \alpha$.
- Let $\xi \leq \alpha_f$ be minimal such that $\operatorname{otp}(G \cap \xi) = \alpha$.
- Apply clause (5) to $\xi$ with $\eta = \tau_f \upharpoonright \xi$, $X = G \cap \xi$, to get $a \subseteq \xi$ of order type $\omega$ with the property given in the clause.
- We wish to use this property of the set $a$ with $b = e$ and $\mathcal{A} = \mathcal{A}_\delta$.
- If we succeed, then we will get a $\zeta < \xi \leq \alpha_f$ such that both $e^0(\tau_f) \upharpoonright \zeta \cap e$ and $e^1(\tau_f) \upharpoonright \zeta \cap e$. 

Bibliography
The case $\alpha < \varsigma$

- So assume that $|G| \geq \alpha$.
- Let $\xi \leq \alpha_f$ be minimal such that $\text{otp}(G \cap \xi) = \alpha$.
- Apply clause (5) to $\xi$ with $\eta = \tau_f \upharpoonright \xi$, $X = G \cap \xi$, to get $a \subset \xi$ of order type $\omega$ with the property given in the clause.
- We wish to use this property of the set $a$ with $b = e$ and $\mathcal{A} = \mathcal{A}_\delta$.
- If we succeed, then we will get a $\zeta < \xi \leq \alpha_f$ such that both $e^0_{(\tau_f) \upharpoonright \zeta} \cap e$ and $e^1_{(\tau_f) \upharpoonright \zeta} \cap e$.
- It suffices to produce a sequence $\langle a_n : \in \omega \rangle$ of distinct elements of $\mathcal{A}_\delta$ and an increasing sequence $\langle k_n : n \in \omega \rangle$ of elements of $\omega$ such that $|e \cap a_n \cap c_{\eta,a,k_n}| < \omega$. 
The case $\alpha < \mathfrak{s}$

- Note that if $a(k) \in X$, then there is $a \in F_{a(k)}$ such that $e \cap a \cap c_{\eta,a,k}$ is infinite.
- By the minimality of $F_{a(k)}$, if $k < l$ and $a(k) \in X$ and $a(l) \in X$, then $F_{a(k)} \cap F_{a(l)} = 0$.
- Since $a \cap X$ is infinite, we are done!
The case $\alpha < \omega$

- Note that if $a(k) \in X$, then there is $a \in F_{a(k)}$ such that $e \cap a \cap c_{\eta,a,k}$ is infinite.
- By the minimality of $F_{a(k)}$, if $k < l$ and $a(k) \in X$ and $a(l) \in X$, then $F_{a(k)} \cap F_{a(l)} = 0$.
- Since $a \cap X$ is infinite, we are done!
- So this contradiction shows that $|G| < \alpha$. So we can find $a_\delta$ and $\eta_\delta$ as needed ($\eta_\delta = \tau_f$).