Constructing special almost disjoint families

Dilip Raghavan

National University of Singapore

Winter School in Abstract Analysis,
International Center for Spiritual Rehabilitation, Hejnice
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Outline

1. A completely separable family from $\mathfrak{s} \leq \mathfrak{a}$
2. A completely separable family from $\mathfrak{c} < \aleph_\omega$

Building a completely separable family

Theorem (Mildenberger, R., and Steprans [1])

If $\mathfrak{s} \leq \mathfrak{a}$, then there is a completely separable family.

- The basic framework is contained in this proof. It is also the simplest.
- Easy to see that a completely separable family exists if $\mathfrak{a} = \mathfrak{c}$.
- (Balcar, Simon, Vojtas): They exist if any one of these holds: $\mathfrak{s} = \omega_1, \mathfrak{b} = \mathfrak{d}$, or $\mathfrak{d} \leq \mathfrak{a}$.
- The hypothesis $\mathfrak{s} \leq \mathfrak{a}$ is weaker than all of the above.

Building a completely separable family

- $\mathcal{F} \subset \mathcal{P}(\omega)$ is said to be (ω, ω)-*splitting* if for each collection $\{b_n : n \in \omega\} \subset [\omega]^\omega$, there exists $a \in \mathcal{F}$ such that $\exists^\infty n \in \omega\, [|a \cap b_n| = \omega]$ and $\exists^\infty n \in \omega\, [|(\omega \setminus a) \cap b_n| = \omega]$.

Definition

$\mathfrak{s}_{\omega,\omega} = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{P}(\omega) \wedge \mathcal{F}$ *is* $(\omega, \omega) -$ *splitting*$\}$.

- Note that $\mathfrak{s} \leq \mathfrak{s}_{\omega,\omega}$ is clear.

Building a completely separable family

Lemma

$\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.

Proof.

Case 1: $\mathfrak{s} < \mathfrak{b}$. Let $\langle e_\alpha : \alpha < \kappa \rangle$ be a splitting family. Suppose it is not (ω, ω)-splitting. Fix $\{b_n : n \in \omega\}$ witnessing this. In other words, for each $\alpha < \kappa$ there is $i_\alpha \in 2$ such that $\forall^\infty n \in \omega \left[\left| b_n \cap e_\alpha^{i_\alpha} \right| < \omega \right]$. WLOG, the b_n are pairwise disjoint. Now, for each $\alpha < \mathfrak{s}$ define $f_\alpha \in \omega^\omega$ as follows:

$$f_\alpha(n) = \begin{cases} \sup(b_n \cap e_\alpha^{i_\alpha}) & \text{if } \left| b_n \cap e_\alpha^{i_\alpha} \right| < \omega \\ 0 & \text{otherwise} \end{cases}$$

⊣

Building a completely separable family

Proof.

By hypothesis the first case occurs for all but finitely many n. Since $\mathfrak{s} < \mathfrak{b}$, find $f \in \omega^\omega$ such that $\forall \alpha < \mathfrak{s}\, [f_\alpha \leq^* f]$. Choose $k_n \in b_n$ such that $k_n > f(n)$. Then $\{k_n : n \in \omega\}$ is an infinite set not split by any e_α.

Building a completely separable family

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Case 2: $\mathfrak{b} \leq \mathfrak{s}$. Proof by picture on the board. ⊣

Building a completely separable family

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Case 2: $\mathfrak{b} \leq \mathfrak{s}$. Proof by picture on the board. ⊣

Lemma

If $\langle e_\alpha : \alpha < \mathfrak{s} \rangle$ is (ω, ω)-splitting, then for any infinite a.d. family $\mathscr{A} \subset [\omega]^\omega$ and for any $b \in \mathcal{I}^+(\mathscr{A})$, there is $\alpha < \mathfrak{s}$ such that $b \cap e_\alpha^0 \in \mathcal{I}^+(\mathscr{A})$ and $b \cap e_\alpha^1 \in \mathcal{I}^+(\mathscr{A})$.

Building a completely separable family

Lemma

Let $\langle e_\alpha : \alpha < \kappa \rangle$ witness $\kappa = \mathfrak{s}_{\omega,\omega}$. Let $\mathscr{A} \subset [\omega]^\omega$ be any a.d. family. Then for each $b \in \mathcal{I}^+(\mathscr{A})$, there is an $\alpha < \kappa$ such that $b \cap e^0_\alpha \in \mathcal{I}^+(\mathscr{A})$ and $b \cap e^1_\alpha \in \mathcal{I}^+(\mathscr{A})$.

Proof.

We may assume that there exist an infinite set $\{a_n : n \in \omega\} \subset \mathscr{A}$ such that $\forall n \in \omega\, [|a_n \cap b| = \omega]$ (otherwise it is easy). Let $\alpha < \kappa$ be such that $\exists^\infty n \in \omega \left[\left|e^0_\alpha \cap a_n \cap b\right| = \omega\right]$ and $\exists^\infty n \in \omega \left[\left|e^1_\alpha \cap a_n \cap b\right| = \omega\right]$. α is as needed. ⊣

Building a completely separable family

- Say $\kappa = \mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ and say $\langle x_\alpha : \alpha < \kappa \rangle$ is an (ω, ω)-splitting family.

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- Construct $\langle a_\alpha : \alpha < \mathfrak{c} \rangle$ and $\langle \sigma_\alpha : \alpha < \mathfrak{c} \rangle \subset 2^{<\kappa}$ such that:
  1. $\forall \alpha < \mathfrak{c} \forall \xi < \mathrm{dom}(\sigma_\alpha) \left[a_\alpha \subset^* x_\xi^{\sigma_\alpha(\xi)} \right]$;
  2. $\forall \alpha < \beta < \mathfrak{c} \left[\sigma_\alpha \neq \sigma_\beta \right]$.
- Observe that if $\alpha \neq \beta$, then by (2), a_α and a_β are a.d. *unless* σ_α and σ_β are comparable.

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- Observe that if $\alpha \neq \beta$, then by (2), a_α and a_β are a.d. *unless* σ_α and σ_β are comparable.
- Main point: At a stage $\delta < \mathfrak{c}$ $\mathscr{A}_\delta = \{a_\alpha : \alpha < \delta\}$ is "nowhere MAD" – i.e. if $b \in \mathcal{I}^+(\{a_\alpha : \alpha < \delta\})$, then there is $a \in [b]^\omega$ such that $\forall \alpha < \delta \, [|a \cap a_\alpha| < \omega]$ (and also a node σ associated with a).

Building a completely separable family

- If $b \in \mathcal{I}^+(\mathscr{A}_\delta)$, then look for least $\alpha_0 < \kappa$ such that $b \cap x^0_{\alpha_0} \in \mathcal{I}^+(\mathscr{A}_\delta)$ and $b \cap x^1_{\alpha_0} \in \mathcal{I}^+(\mathscr{A}_\delta)$.
- There is a unique $\tau_0 \in 2^{\alpha_0}$ such that

$$\forall \xi < \alpha_0 \forall i \in 2 \left[\tau_0(\xi) = i \leftrightarrow b \cap x^i_\xi \in \mathcal{I}^+(\mathscr{A}_\delta) \right].$$

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$$\forall \xi < \alpha_0 \forall i \in 2 \left[\tau_0(\xi) = i \leftrightarrow b \cap x^i_\xi \in \mathcal{I}^+(\mathscr{A}_\delta)\right].$$

- Proceeding in the same way one can build two sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset \kappa$ and $\langle \tau_s : s \in 2^{<\omega} \rangle \subset 2^{<\kappa}$ such that:
  (3) $\forall s \in 2^{<\omega} \forall i \in 2 \left[\alpha_s = \mathrm{dom}(\tau_s) \wedge \alpha_{s^\frown\langle i \rangle} > \alpha_s \wedge \tau_{s^\frown\langle i \rangle} \supset \tau_s{}^\frown\langle i \rangle\right]$;
  (4) for each $s \in 2^{<\omega}$ and for each $\xi < \alpha_s$, $x^{1-\tau_s(\xi)}_\xi \cap b \cap \left(\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}\right) \in \mathcal{I}(\mathscr{A}_\delta)$;
  (5) for each $s \in 2^{<\omega}$, both $x^0_{\alpha_s} \cap b \cap \left(\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}\right) \in \mathcal{I}^+(\mathscr{A}_\delta)$ and $x^1_{\alpha_s} \cap b \cap \left(\bigcap_{t \subsetneq s} x^{\tau_s(\alpha_t)}_{\alpha_t}\right) \in \mathcal{I}^+(\mathscr{A}_\delta)$.

Building a completely separable family

- For each $f \in 2^\omega$, put $\alpha_f = \sup \left\{ \alpha_{(f \restriction n)} : n \in \omega \right\}$ and $\tau_f = \bigcup_{n \in \omega} \tau_{(f \restriction n)}$.
- Note $\alpha_f < \kappa$.

Building a completely separable family

- For each $f \in 2^\omega$, put $\alpha_f = \sup\left\{\alpha_{(f \restriction n)} : n \in \omega\right\}$ and $\tau_f = \bigcup_{n \in \omega} \tau_{(f \restriction n)}$.
- Note $\alpha_f < \kappa$.
- Find $f \in 2^\omega$ such that $\tau_f \notin \{\sigma \in 2^{<\kappa} : \exists \alpha < \delta\, [\sigma \subset \sigma_\alpha]\}$.
- $e \in [b]^\omega \cap \mathcal{I}^+(\mathscr{A}_\delta)$ such that $\forall n \in \omega\, [e \subset^* e_n]$, where $e_n = b \cap \left(\bigcap_{m<n} x_{\alpha_{(f \restriction m)}}^{\tau_f(\alpha_{(f \restriction m)})}\right)$.

Building a completely separable family

- For any $\xi < \alpha_f$, there is $F_\xi \in [\delta]^{<\omega}$ such that

$$\left(x_\xi^{1-\tau_f(\xi)} \cap e\right) \subset^* \left(\bigcup_{\alpha \in F_\xi} a_\alpha\right).$$

- Consider $\mathcal{F} = \bigcup_{\xi < \alpha_f} F_\xi$ and $\mathcal{G} = \{\alpha < \delta : \sigma_\alpha \subset \tau_f\}$.
- $|\mathcal{F} \cup \mathcal{G}| < \kappa \leq \mathfrak{a}$.

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- $|\mathcal{F} \cup \mathcal{G}| < \kappa \leq \mathfrak{a}$.
- So there is $a \in [e]^\omega$ such that $\forall \alpha \in \mathcal{F} \cup \mathcal{G}\, [|a \cap a_\alpha| < \omega]$.

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- Consider $\mathcal{F} = \bigcup_{\xi < \alpha_f} F_\xi$ and $\mathcal{G} = \{\alpha < \delta : \sigma_\alpha \subset \tau_f\}$.
- $|\mathcal{F} \cup \mathcal{G}| < \kappa \leq \mathfrak{a}$.
- So there is $a \in [e]^\omega$ such that $\forall \alpha \in \mathcal{F} \cup \mathcal{G} \, [|a \cap a_\alpha| < \omega]$.
- Now a and σ_f are as needed:
    1. If $\alpha \in \mathcal{G}$, then a and a_α are a.d. by choice.
    2. If $\alpha \notin \mathcal{G}$, then a_α and a are a.d. because $\forall \xi < \alpha_f \left[a \subset^* x_\xi^{\sigma_f(\xi)}\right]$.

The case $\mathfrak{a} < \mathfrak{s}$

- When $\mathfrak{a}$ is small, $\mathfrak{b}$ is also small.
- Key point: there is a small collection of sets that splits any set of a specific form (even though there are no small splitting families).

The case $\mathfrak{a} < \mathfrak{s}$

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- Key point: there is a small collection of sets that splits any set of a specific form (even though there are no small splitting families).

Lemma

Let $\langle c_n : n \in \omega \rangle$ be pairwise disjoint elements of $[\omega]^\omega$. Then there is a collection $\langle x_\alpha : \alpha < \mathfrak{b} \rangle \subset \mathcal{P}(\omega)$ such that for any $b \in [\omega]^\omega$ and any infinite a.d. family $\mathscr{A} \subset [\omega]^\omega$, if for all $n \in \omega$ and for all $f \in \omega^\omega$, $\bigcup_{m \geq n} \{k \in b \cap c_m : k > f(m)\} \in \mathcal{I}^+(\mathscr{A})$, then there is $\alpha < \mathfrak{b}$ such that $x_\alpha^0 \cap b \in \mathcal{I}^+(\mathscr{A})$ and $x_\alpha^1 \cap b \in \mathcal{I}^+(\mathscr{A})$.

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Proof.

Fix a $<^*$-increasing everywhere unbounded family $\langle f_\alpha : \alpha < \mathfrak{b} \rangle \subset \omega^\omega$. For each $\alpha < \mathfrak{b}$ and $n \in \omega$, let $x_{\alpha,n} = \{k \in c_n : k \leq f_\alpha(n)\}$. Let $x_\alpha = \bigcup_{n\in\omega} x_{\alpha,n}$. Why does this work?

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Case I: there is $e \in [d]^\omega$ which is a.d. from every $a \in \mathscr{A}$. Let $X = \{m \in \omega : e \cap c_m \neq 0\}$. Define $f : X \to \omega$ by $f(m) = \min(e \cap c_m)$. There is $\alpha < \mathfrak{b}$ such that $\exists^\infty m \in X\, [f(m) \leq f_\alpha(m)]$. For any such $m \in X$, $x_{\alpha,m} \cap e \neq 0$. So $\left|x^0_\alpha \cap e\right| = \omega$. This implies $x^0_\alpha \cap d$, and hence $x^0_\alpha \cap b$ are in $\mathcal{I}^+(\mathscr{A})$. On the other hand, $x^1_\alpha \cap b \in \mathcal{I}^+(\mathscr{A})$ by hypothesis. ⊣

The case $\mathfrak{a} < \mathfrak{s}$

Proof.

Case II: there are infinitely many $a \in \mathscr{A}$ such that $|a \cap d| = \omega$. Fix such a family $\{a_n : n \in \omega\} \subset \mathscr{A}$. For each $n \in \omega$, let $X_n = \{m \in \omega : a_n \cap d \cap c_m \neq 0\}$. There is $\alpha < \mathfrak{b}$ such that for each $n \in \omega$, $\exists^\infty m \in X_n \left[c_m \cap d \cap a_n \cap (f_\alpha(m) + 1) \neq 0\right]$. Then for each $n \in \omega$, $\left|a_n \cap d \cap x^0_\alpha\right| = \omega$. So $d \cap x^0_\alpha$ and hence $b \cap x^0_\alpha$ are in $\mathcal{I}^+(\mathscr{A})$. $x^1_\alpha \cap b$ is in $\mathcal{I}^+(\mathscr{A})$ by hypothesis. ⊣

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- Solution: make the tree more complicated.

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- There is a problem: the collection $\langle c_n : n \in \omega \rangle$ that we care about will keep changing at every stage of the construction.
- Solution: make the tree more complicated.
- Main difference: instead of using a sequence of sets $\langle e_\alpha : \alpha < \kappa \rangle$, use a tree of sets $\langle e_\eta : \eta \in 2^{<\kappa} \rangle$.
- The pair e^0, e^1 used at a node of the tree now depends not just on the height of that node, but also on all the pairs of sets that occur below that node.

The case $\mathfrak{a} < \mathfrak{s}$

- Along each (long enough) branch ψ of the tree, each countable subset of ψ can be "captured" at some node η that lies on ψ.
- This "captured" countable set determines a collection $\langle c_n : n \in \omega \rangle$.
- The sets that hit infinitely many of the c_n will be split using a small family *before* ψ is reached.

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- This "captured" countable set determines a collection $\langle c_n : n \in \omega \rangle$.
- The sets that hit infinitely many of the c_n will be split using a small family *before* ψ is reached.
- The assumption that $\mathfrak{s} < \aleph_\omega$ becomes relevant for capturing the countable sets.

The case $\mathfrak{a} < \mathfrak{s}$

Definition

Let κ be any cardinal. A set $X \subset [\kappa]^{\leq\omega}$ is called cofinal if $\forall a \in [\kappa]^{\leq\omega} \exists b \in X \, [a \subset b]$.

$$\mathrm{cf}(\langle [\kappa]^{\leq\omega}, \subset \rangle) = \min \left\{ |X| : X \subset [\kappa]^{\leq\omega} \text{ is cofinal} \right\}.$$

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$$\mathrm{cf}(\langle [\kappa]^{\leq\omega}, \subset \rangle) = \min\left\{ |X| : X \subset [\kappa]^{\leq\omega} \text{ is cofinal} \right\}.$$

- For any $n < \omega$, $\mathrm{cf}(\langle [\aleph]^{\leq\omega}, \subset \rangle) = \aleph_n$ (obvious for $\aleph_1$; by induction for larger n).
- So for any $n < \omega$, there is a sequence $\langle u_\alpha : \omega \leq \alpha < \aleph_n \rangle$ such that
  1. $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$;
  2. if $X \subset \aleph_n$ is any uncountable set, then there exists $\alpha < \sup(X)$ such that $[|u_\alpha \cap X| = \omega]$.
- If in addition you know that $\mathfrak{b} \leq \aleph_n$, then you can strengthen 1 to say that $\mathrm{otp}(u_\alpha) = \omega$; but 2 will only apply to sets order type at least $\mathfrak{b}$.

The case $\mathfrak{a} < \mathfrak{s}$

Definition

For cardinals $\kappa > \lambda > \omega$, $P(\kappa, \lambda)$ says that there is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ such that

1. $u_\alpha \subset \alpha$ *and* $|u_\alpha| = \omega$
2. *for each $X \subset \kappa$, if X is bounded in κ and* $\operatorname{otp}(X) = \lambda$, *then* $\exists \omega \leq \alpha < \sup(X) \, [|u_\alpha \cap X| = \omega]$.

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1. $u_\alpha \subset \alpha$ *and* $|u_\alpha| = \omega$
2. *for each $X \subset \kappa$, if X is bounded in κ and $\operatorname{otp}(X) = \lambda$, then* $\exists \omega \leq \alpha < \sup(X) [|u_\alpha \cap X| = \omega]$.

- If $\mathfrak{b} \leq \lambda < \kappa < \aleph_\omega$, then $P(\kappa, \lambda)$ is true.

Theorem (Shelah, 2010 [2])

If $\mathfrak{a} < \mathfrak{s}$ and $P(\mathfrak{s}, \mathfrak{a})$ holds, then there is a completely separable family.

- Forcing the failure of the hypothesis needs large cardinals (and unknown if $\mathfrak{a} > \omega_1$).

The case $\mathfrak{a} < \mathfrak{s}$

- At a stage $\delta < \mathfrak{c}$ we have $\mathscr{A}_\delta = \langle a_\alpha : \alpha < \delta \rangle$, a subtree $\mathcal{T}_\delta \subset 2^{<\mathfrak{s}}$, a labeling $\langle e_\eta : \eta \in \mathcal{T}_\delta \rangle$, and a sequence of nodes $\langle \eta_\alpha : \alpha < \delta \rangle \subset \mathcal{T}_\delta$ such that for each $\alpha < \delta$:
  1. $\forall \xi \in \operatorname{dom}(\eta_\alpha) \left[a_\alpha \subset^* e^{\eta_\alpha(\xi)}_{\eta_\alpha \restriction \xi} \right]$;

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  1. $\forall \xi \in \mathrm{dom}(\eta_\alpha) \left[a_\alpha \subset^* e^{\eta_\alpha(\xi)}_{\eta_\alpha \restriction \xi} \right]$;
  2. if $\sigma \in 2^{\mathfrak{s}}$ and if $\sigma \restriction \xi \in \mathcal{T}_\delta$ for all $\xi < \mathfrak{s}$, then $\{e_{\sigma \restriction \xi} : \xi < \mathfrak{s}\}$ is an (ω, ω)-splitting family;

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  1. $\forall \xi \in \mathrm{dom}(\eta_\alpha) \left[a_\alpha \subset^* e_{\eta_\alpha \restriction \xi}^{\eta_\alpha(\xi)} \right]$;
  2. if $\sigma \in 2^\mathfrak{s}$ and if $\sigma \restriction \xi \in \mathcal{T}_\delta$ for all $\xi < \mathfrak{s}$, then $\{ e_{\sigma \restriction \xi} : \xi < \mathfrak{s} \}$ is an (ω, ω)-splitting family;
  3. $|\mathcal{T}_\delta| < \mathfrak{c}$ (more precisely $\mathcal{T}_\delta$ is the union of $< \mathfrak{c}$ chains) and $e_{\eta_\alpha} = a_\alpha$;

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  1. $\forall \xi \in \mathrm{dom}(\eta_\alpha) \left[a_\alpha \subset^* e^{\eta_\alpha(\xi)}_{\eta_\alpha \restriction \xi} \right]$;
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  3. $|\mathcal{T}_\delta| < \mathfrak{c}$ (more precisely $\mathcal{T}_\delta$ is the union of $< \mathfrak{c}$ chains) and $e_{\eta_\alpha} = a_\alpha$;
  4. For $\xi < \mathfrak{s}$, $\eta \in 2^\xi \cap \mathcal{T}_\delta$, a set $a \subset \xi$ of order type ω, and $n \in \omega$, we use the notation $c_{\eta,a,n} = \left(\bigcap_{m<n} e^{\eta(a(m))}_{\eta \restriction a(m)} \right) \cap e^{1-\eta(a(n))}_{\eta \restriction a(n)}$, where $a(m)$ denotes the mth element of a;
  5. If $\xi < \mathfrak{s}$ and if $X \subset \xi$ has order type $\mathfrak{a}$ and $\sup(X) = \xi$, then for any $\eta \in 2^\xi \cap \mathcal{T}_\delta$, there is $a \subset \xi$ with $\mathrm{otp}(a) = \omega$ such that $|a \cap X| = \omega$ and for every $b \in [\omega]^\omega$ and any infinite a.d. family $\mathscr{A} \subset [\omega]^\omega$, if for all $n \in \omega$ and for all $f \in \omega^\omega$, $\bigcup_{m \ge n} \{k \in b \cap c_m : k > f(m)\} \in \mathcal{I}^+(\mathscr{A})$, then there is $\zeta < \xi$ such that $b \cap e^0_{\eta \restriction \zeta} \in \mathcal{I}^+(\mathscr{A})$ and $b \cap e^1_{\eta \restriction \zeta} \in \mathcal{I}^+(\mathscr{A})$.

The case $\mathfrak{a} < \mathfrak{s}$

- Given $\langle u_\alpha : \alpha < \mathfrak{s} \rangle$ witnessing $P(\mathfrak{s}, \mathfrak{a})$, a family $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ witnessing $\mathfrak{b} < \mathfrak{s}$, and an (ω, ω)-splitting family $\langle x_\alpha : \alpha < \mathfrak{s} \rangle$, arranging (1)-(5) is just a matter of bookkeeping.
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- Details of the bookkeeping are not deep (just messy).
- the idea is that along a branch η, any subset X of order type $\mathfrak{a}$ will be “trapped” by some u_α.
- This u_α determines a collection $\{c_{\eta, u_\alpha, n} : n \in \omega\}$.
- Together with $\langle f_\beta : \beta < \mathfrak{b} \rangle$ this gives a family $\{y_\beta : \beta < \mathfrak{b}\}$ such that any b that behaves like in the lemma w.r.t. the $c_{\eta, u_\alpha, n}$ is split by one of the y_β.
- There is enough space to enumerate the $\{y_\beta : \beta < \mathfrak{b}\}$ (note: this set does not depend on X) along η; so every b that intersects infinitely many of the $c_{\eta, u_\alpha, n}$ will be split before η is reached.

The case $\mathfrak{a} < \mathfrak{s}$

- At a stage $\delta < \mathfrak{c}$, fix some $b \in \mathcal{I}^+(\mathscr{A}_\delta)$
- By clause (2), we can once again build sequences $\langle \alpha_s : s \in 2^{<\omega} \rangle \subset \mathfrak{s}$ and $\langle \tau_s : s \in 2^{<\omega} \rangle \subset \mathcal{T}_{\delta+1}$ as before.
- As before, for any $f \in 2^\omega$, if $\tau_f = \bigcup_{n\in\omega} \tau_{(f\restriction n)}$ and if $\alpha_f = \sup \left\{ \alpha_{(f\restriction n)} : n \in \omega \right\}$, then $\alpha_f < \mathfrak{s}$, and $b \cap e_{\tau_{(f\restriction 0)}}^{\tau_f(\alpha_{(f\restriction 0)})} \supset b \cap e_{\tau_{(f\restriction 0)}}^{\tau_f(\alpha_{(f\restriction 0)})} \cap e_{\tau_{(f\restriction 1)}}^{\tau_f(\alpha_{(f\restriction 1)})} \supset \cdots$ is a decreasing sequence of sets in $\mathcal{I}^+(\mathscr{A}_\delta)$.

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- Choose $f \in 2^\omega$ such that $\tau_f \notin \mathcal{T}_\delta$ and choose $e \in [b]^\omega \cap \mathcal{I}^+(\mathscr{A}_\delta)$ that is almost included in this decreasing sequence.

The case $\mathfrak{a} < \mathfrak{s}$

- As before, for any $\xi < \alpha_f$, there is a *minimal* $F_\xi \in [\delta]^{<\omega}$ such that $e \cap e^{1-\tau_f(\xi)}_{(\tau_f)\restriction\xi} \subset^* \bigcup_{\alpha \in F_\xi} a_\alpha$.
- Recall clause (3) which says that for any $\alpha < \delta$, $e_{\eta_\alpha} = a_\alpha$.
- For any $\alpha < \delta$, if $\eta_\alpha \subset \tau_f$, then $\mathrm{dom}(\eta_\alpha) < \alpha_f$ and $\tau_f(\mathrm{dom}(\eta_\alpha)) = 1$ because of this clause.

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- Conclusion: It is enough to find $a \in [e]^\omega$ such that $\forall \xi < \alpha_f \left[a \subset^* e^{\tau_f(\xi)}_{(\tau_f)\restriction\xi} \right]$.

The case $\mathfrak{a} < \mathfrak{s}$

- Consider the collection G of all $\zeta < \alpha_f$ for which there is $x \in [e]^\omega$ such that:
  1. $\forall \xi < \zeta \left[x \subset^* e^{\tau_f(\xi)}_{(\tau_f)\restriction\xi} \right]$;
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  2. $x \cap e^{1-\tau_f(\zeta)}_{(\tau_f)\restriction\zeta}$ is infinite.
- If $|G| < \mathfrak{a}$, then we can find $a \in [e]^\omega$ as needed.
- Why? $\left|\bigcup_{\zeta \in G} F_\zeta\right| < \mathfrak{a}$. Take $a \in [e]^\omega$ which is a.d. from everything in $\bigcup_{\zeta \in G} F_\zeta$.
- Suppose there exists $\zeta < \alpha_f$ such that $\left|a \cap e^{1-\tau_f(\zeta)}_{(\tau_f)\restriction\zeta}\right| = \omega$. Take the least such ζ. Then a witnesses that $\zeta \in G$, which is a contradiction.

The case $\mathfrak{a} < \mathfrak{s}$

- So assume that $|G| \geq \mathfrak{a}$.
- Let $\xi \leq \alpha_f$ be minimal such that $\mathrm{otp}(G \cap \xi) = \mathfrak{a}$.
- Apply clause (5) to ξ with $\eta = \tau_f \restriction \xi$, $X = G \cap \xi$, to get $a \subset \xi$ of order type ω with the property given in the clause.
- We wish to use this property of the set a with $b = e$ and $\mathscr{A} = \mathscr{A}_\delta$.

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- If we succeed, then we will get a $\zeta < \xi \leq \alpha_f$ such that both $e^0_{(\tau_f)\restriction\zeta} \cap e$ and $e^1_{(\tau_f)\restriction\zeta} \cap e$.

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- We wish to use this property of the set a with $b = e$ and $\mathscr{A} = \mathscr{A}_\delta$.
- If we succeed, then we will get a $\zeta < \xi \leq \alpha_f$ such that both $e^0_{(\tau_f)\restriction\zeta} \cap e$ and $e^1_{(\tau_f)\restriction\zeta} \cap e$.
- It suffices to produce a sequence $\langle a_n :\in \omega\rangle$ of distinct elements of $\mathscr{A}_\delta$ and an increasing sequence $\langle k_n : n \in \omega\rangle$ of elements of ω such that $\left|e \cap a_n \cap c_{\eta,a,k_n}\right| < \omega$.

The case $\mathfrak{a} < \mathfrak{s}$

- Note that if $a(k) \in X$, then there is $a \in F_{a(k)}$ such that $e \cap a \cap c_{\eta,a,k}$ is infinite.
- By the minimality of $F_{a(k)}$, if $k < l$ and $a(k) \in X$ and $a(l) \in X$, then $F_{a(k)} \cap F_{a(l)} = 0$.
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- Since $a \cap X$ is infinite, we are done!
- So this contradiction shows that $|G| < \mathfrak{a}$. So we can find a_δ and η_δ as needed ($\eta_\delta = \tau_f$).

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