Constructing special almost disjoint families

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Outline

1. Definitions and motivations
2. Recent Progress
3. Some questions
We say that two infinite subsets $a$ and $b$ of $\omega$ are *almost disjoint or a.d.* if $a \cap b$ is finite.

We say that a family $\mathcal{A} \subset [\omega]^{\omega}$ is *almost disjoint or a.d.* if its members are pairwise almost disjoint.

A *Maximal Almost Disjoint family, or MAD family* is an infinite a.d. family that is not properly contained in a larger a.d. family.

Equivalently, an infinite a.d. family $\mathcal{A} \subset [\omega]^{\omega}$ is MAD iff

$$\forall b \in [\omega]^{\omega} \exists a \in \mathcal{A} \ [|b \cap a| = \omega].$$
By Zorn’s Lemma, any infinite a.d. family can be extended to a MAD family.

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For example the size of $\mathcal{A}$.

If we want to make $|\mathcal{A}|$ as large as possible, then we can, but we need an intermediate step.
By Zorn’s Lemma, any infinite a.d. family can be extended to a MAD family.

This construction usually doesn’t allow us to control other combinatorial properties of $\mathcal{A}$.

For example the size of $\mathcal{A}$.

If we want to make $|\mathcal{A}|$ as large as possible, then we can, but we need an intermediate step.

Identify $\omega$ with $2^{<\omega}$. Then the branches form an a.d. family of size $\mathfrak{c}$. Extend it to a MAD family.
Definitions and motivations

- How small can a MAD family be?

Definition

\[ a = \min \{ |A| : A \subset \omega \text{ and } A \text{ is a MAD family} \} \]

The value of \( a \) is not decided in ZFC. There are several such cardinal invariants that play a crucial role in many combinatorial constructions. Usually take the form of the least size of a family of a certain sort.
Definitions and motivations

- How small can a MAD family be?

**Definition**

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- The value of \( \alpha \) is not decided in ZFC.
- There are several such *cardinal invariants*.
- Play a crucial role in many combinatorial constructions.
- Usually take the form of the least size of a family of a certain sort.
Definitions and motivations

- for $a, b \in \mathcal{P}(\omega)$, $a$ splits $b$ if $|a \cap b| = |(\omega \setminus a) \cap b| = \omega$.
- $F \subset \mathcal{P}(\omega)$ is called a splitting family if $\forall b \in [\omega]^\omega \exists a \in F \ [a \text{ splits } b]$. 
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**Definition**

\[ s = \min \{|F| : F \subset \mathcal{P}(\omega) \text{ and } F \text{ is a splitting family} \}. \]
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$s = \min\{|F| : F \subset \mathcal{P}(\omega) \text{ and } F \text{ is a splitting family}\}$.

- A family $F \subset \omega^{\omega}$ is called unbounded if it has no upper bound in $\langle \omega^{\omega}, \leq^{*} \rangle$.
- $F \subset \omega^{\omega}$ is called dominating if it is cofinal in $\langle \omega^{\omega}, \leq^{*} \rangle$. 
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Definition

$b = \min\{|F| : F \subset \omega^{\omega} \text{ is an unbounded family}\}$.

$d = \min\{|F| : F \subset \omega^{\omega} \text{ is a dominating family}\}$. 
Definitions and motivations

Definition

For any family $\mathcal{A} \subset \mathcal{P}(\omega)$, the ideal generated by $\mathcal{A}$ (together with the Fréchet ideal) is denoted by $I(\mathcal{A})$.

Definition

For any ideal $I$ on $\omega$, $I^+$ denotes $\mathcal{P}(\omega) \setminus I$. The sets in $I^+$ are called $I$-positive. $I^*$ denotes $\{\omega \setminus a : a \in I\}$, this is the dual filter to $I$. An ideal $I$ is said to be tall if $\forall b \in [\omega]^\omega \exists a \in [b]^\omega [a \in I]$. 
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- We are interested in almost disjoint families for which $I(\mathcal{A})$ enjoys certain strong properties.
Definitions and motivations

- If $\mathcal{A}$ is a.d., then $I^+(\mathcal{A})$ always has a strong combinatorial property.

Theorem

If $\mathcal{A} \subset \mathcal{P}(\omega)$ is an infinite a.d. family, then $I^+(\mathcal{A})$ is a selective co-ideal.
Definitions and motivations

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**Theorem**

If $\mathcal{A} \subset \mathcal{P}(\omega)$ is an infinite a.d. family, then $I^+(\mathcal{A})$ is a selective co-ideal.

- This essentially means that $I^*(\mathcal{A})$ “can be” extended to a Ramsey ultrafilter.

**Definition**

$I^+$ is called a selective coideal if for every sequence $e_0 \supset e_1 \supset \cdots$, with $e_i \in I^+$, there is an $e = \{n_0 < n_1 < \cdots\} \in I^+$ such that $n_0 \in e_0$ and $n_{i+1} \in e_n$ for each $i$.
The main point is the following:

**Lemma**

Suppose $\mathcal{A}$ is an a.d. family. Suppose $b \subset \omega$ and $\exists^\infty a \in \mathcal{A} \ [|a \cap b| = \omega]$. Then $b \in \mathcal{I}^+(\mathcal{A})$. 

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Proof.

If $b \in I(\mathcal{A})$, then there exist $a_0, \ldots a_k \in \mathcal{A}$ such that $b \subset^* a_0 \cup \cdots \cup a_k$. By hypothesis, there is $a \in \mathcal{A} \setminus \{a_0, \ldots, a_k\}$ such that $a \cap b$ is infinite. However $a \cap b$ is a.d. from $a_0 \cup \cdots \cup a_k$ and yet $a \cap b \subset b \subset^* a_0 \cup \cdots \cup a_k$. This is a contradiction. $\square$
Definitions and motivations

- We are interested in families where there is a strong combinatorial relationship between $\mathcal{A}$ and $I^+(\mathcal{A})$.

- A typical example is the following:

**Definition**

An almost disjoint family $\mathcal{A}$ is tight (also called $\aleph_0$-MAD) if for any $\{b_n : n \in \omega\} \subset I^+(\mathcal{A})$, there is $a \in \mathcal{A}$ such that $\forall n \in \omega \ [|a \cap b_n| = \aleph_0]$.

- This asks for a $\sigma$-version of maximality.

- It is also connected with the notion of indestructible MAD families.

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Constructing special almost disjoint families
Definitions and motivations

**Definition**

Let $\mathbb{P}$ be a notion of forcing. A MAD family $\mathbb{A} \subset [\omega]^\omega$ is called $\mathbb{P}$-indestructible if $\Vdash_{\mathbb{P}} \mathbb{A}$ is MAD.

Obviously, if $\mathbb{P}$ does not add reals, then every MAD $\mathbb{A}$ is $\mathbb{P}$-indestructible.

If a MAD $\mathbb{A} \subset [\omega]^\omega$ is indestructible for any $\mathbb{P}$ that adds a real, then $\mathbb{A}$ is also Sacks indestructible.

**Theorem**

Every tight a.d. family is Cohen-indestructible. If a MAD family $\mathbb{A}$ is Cohen-indestructible, then for some $X \in I^+(\mathbb{A})$, $\mathbb{A} \restriction X = \{ X \cap a : a \in \mathbb{A} \}$ is tight.
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- Obviously, if $\mathbb{P}$ does not add reals, then every MAD $\mathcal{A}$ is $\mathbb{P}$-indestructible.
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Every tight a.d. family is Cohen-indestructible. If a MAD family $\mathcal{A}$ is Cohen-indestructible, then for some $X \in \mathcal{I}^+(\mathcal{A})$, $\mathcal{A} \upharpoonright X = \{X \cap a : a \in \mathcal{A}\}$ is tight.
Definitions and motivations

Definition

An a. d. family $\mathcal{A}$ is called weakly tight if for all $\{b_n : n \in \omega\} \subseteq I^+(\mathcal{A})$, there is $a \in \mathcal{A}$ such that $\exists \infty n \in \omega [|a \cap b_n| = \aleph_0]$.

- This is a natural weakening of tight investigated by [1].
- It is connected to the Katetov order on a.d. families.
Definitions and motivations

Definition

An a.d. family $\mathcal{A}$ is called Laflamme if $\mathcal{A}$ is not contained in any $F_\sigma$ ideal on $\omega$.

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Theorem

If $I$ is any $F_\sigma$ ideal on $\omega$, then there is a proper $\omega^\omega$-bounding forcing $\mathbb{P}_I$ which adds an element of $[\omega]^\omega$ that is almost disjoint from every element of $\mathbb{V} \cap I$. 
Laflamme’s questions is related to the problem of whether $d = \aleph_1$ implies $\alpha = \aleph_1$.
If you can get all MAD families to be contained in $F_\sigma$ ideals, then you could hope to increase $\alpha$ without increasing $d$.
We will see that when $d = \aleph_1$, Laflamme families exist.
Definitions and motivations

Definition

An a. d. family is called completely separable if \( \forall b \in I^+(\mathcal{A}) \exists a \in \mathcal{A} \ [a \subset b] \).
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- This question has a long history. It is connected with the existence of ADRs.

Definition

Given \( C \subset [\omega]^\omega \), we say that a family \( \mathcal{A} = \{a_c : c \in C\} \subset [\omega]^\omega \) is an almost disjoint refinement (ADR) of \( C \) if

1. \( \forall c \in C \ [a_c \subset c] \)
2. \( \forall c_0, c_1 \in C \ [c_0 \neq c_1 \implies |a_{c_0} \cap a_{c_1}| < \omega] \).
Definitions and motivations

Fact

Some facts:

- If $C \subset [\omega]^{\omega}$ has an ADR, then there is tall ideal $I$ such that $I \cap C = 0$.
- $I^+$ has an ADR for every tall $I$ iff for every tall $I$ there is a completely separable $\mathcal{A} \subset I$.
- If $\mathcal{A}$ is completely separable, then for every $b \in I^+(\mathcal{A})$, there are $\omega$ many $a \in \mathcal{A}$ such that $a \subset b$. 
Definitions and motivations

Basic Question

*When do these a. d. families exist? Do any of them exist in ZFC?*
Definitions and motivations

Basic Question

*When do these a. d. families exist? Do any of them exist in ZFC?*

- They all exist under CH.
- In these talks we will first survey some of the recent progress on proving existence.
- Then we focus on completely separable and on weakly tight families.
- Both types of families exist if $\mathfrak{c} < \aleph_\omega$ (full proofs, time permitting).
Theorem (Shelah[3], 2010)

If $\mathfrak{c} < \aleph_\omega$, then there is a completely separable a. d. family.
Theorem (Shelah[3], 2010)

If $c < \aleph_\omega$, then there is a completely separable a. d. family.

The proof is in 3 cases:

1. $s < a$
2. $s = a + a$ certain PCF-type assumption holds.
3. $a < s + a$ different PCF-type assumption holds.
Theorem (Shelah[3], 2010)

If \( c < \aleph_\omega \), then there is a completely separable a. d. family.

- The proof is in 3 cases:
  1. \( s < \alpha \)
  2. \( s = \alpha + \) a certain PCF-type assumption holds.
  3. \( \alpha < s + \) a different PCF-type assumption holds.

- The PCF type assumptions both automatically hold if \( c < \aleph_\omega \).
- This proof is the basis for all the recent progress.
Recent progress

- The PCF assumption can be eliminated from case 2 of Shelah’s construction.

**Theorem (Mildenberger, R., and Steprans)**

*If $s \leq \alpha$, then there is a completely separable MAD family.*

- The main point in this proof is that $s = s_{\omega, \omega}$.
Recent progress

Theorem (R. and Steprans)

If $s \leq b$, then there is a weakly tight family.
Recent progress

Theorem (R. and Steprans)

If $s \leq b$, then there is a weakly tight family.

I recently improved this to

Theorem (R.)

If $c < \aleph_\omega$, then there is a weakly tight family.

- The proof is broken down into 2 analogous cases:
  1. $s \leq b$
  2. $b < s + \text{a certain PCF type assumption.}$

- Again the PCF type assumption is automatically satisfied if $c < \aleph_\omega$. 
Let us say that a family $\mathcal{F} \subset \mathcal{P}(\omega)$ is $F_\sigma$ splitting if for each $F_\sigma$ ideal $\mathcal{I}$ on $\omega$, there exists $a \in \mathcal{F}$ such that both $a$ and $\omega \setminus a$ are in $\mathcal{I}^+$. 

**Definition**

$s(\mathcal{F}_\sigma) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{P}(\omega) \text{ is an } F_\sigma \text{-splitting family}\}$. 
Let us say that a family $\mathcal{F} \subset \mathcal{P}(\omega)$ is $F_\sigma$ splitting if for each $F_\sigma$ ideal $\mathcal{I}$ on $\omega$, there exists $a \in \mathcal{F}$ such that both $a$ and $\omega \setminus a$ are in $\mathcal{I}^+$.

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**Definition**

For a filter $\mathcal{F}$ on $\omega$, let

$p(\mathcal{F}) = \{|X| : X \subset \mathcal{F} \text{ and } X \text{ does not have a pseudointersection in } \mathcal{F}^+\}$.
Let us say that a family $\mathcal{F} \subset P(\omega)$ is $F_\sigma$ splitting if for each $F_\sigma$ ideal $\mathcal{I}$ on $\omega$, there exists $a \in \mathcal{F}$ such that both $a$ and $\omega \setminus a$ are in $\mathcal{I}^+$. 

**Definition**

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$$p(\mathcal{F}_\sigma) = \min\{p(\mathcal{F}) : \mathcal{F} \text{ is a tall } F_\sigma \text{ -- filter}\}.$$
Recent progress

- $p(\mathcal{F}_\sigma)$ is consistently bigger than $\mathfrak{d}$.
- $\text{add}(\mathcal{N}) \leq p(\mathcal{F}_\sigma)$
- $s(\mathcal{F}_\sigma) \leq \min\{\max\{b, s\}, \text{non}(\mathcal{N})\}$.
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**Theorem (R.)**

1. If $s(F_\sigma) \leq p(F_\sigma)$, then there is a Laflamme family.
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Theorem (R.)

1. If $s(F_\sigma) \leq p(F_\sigma)$, then there is a Laflamme family.
2. If $b \leq p(F_\sigma) < \aleph_\omega$, then there is a Laflamme family.
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**Theorem (R.)**

1. If $s(F_\sigma) \leq p(F_\sigma)$, then there is a Laflamme family.
2. If $b \leq p(F_\sigma) < \aleph_\omega$, then there is a Laflamme family.

- There are 2 cases:
  1. $s(F_\sigma) \leq p(F_\sigma)$.
  2. $b \leq p(F_\sigma) + \text{a PCF-type assumption}$. 
Recent progress

Corollary

1. If $b = s = \aleph_1$, then there is a Laflamme family.
2. If $\text{non}(\mathcal{N}) = \aleph_1$, then there is a Laflamme family.
Questions

Question

Is there a Laflamme family assuming $\mathfrak{c} < \aleph_\omega$?

- What is still open is the case: $\mathfrak{p}(\mathcal{F}_\sigma) < \min\{b, s(\mathcal{F}_\sigma)\}$.
- An interesting sub-question is what happens when $b = \mathfrak{c}$?
A MAD family $\mathcal{A} \subset [\omega]^{\omega}$ is Sacks indestructible iff for each 1-1 map $\Sigma : 2^{<\omega} \to \omega$, there exists $a \in \mathcal{A}$ such that
\[ \exists f \in 2^{\omega} \left[ |a \cap (\Sigma'' \{f \upharpoonright n : n \in \omega\})| = \omega \right]. \]

If $a < c$, then any MAD family of size $a$ is Sacks indestructible. So you can assume $a = c$ for free.
Question

*Can the general method be modified to construct MAD families in $\omega^\omega$ with special properties?*
