# Constructing special almost disjoint families

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# Outline

1. Definitions and motivations
2. Recent Progress
3. Some questions

# Definitions and motivations

- We say that two infinite subsets $a$ and $b$ of $\omega$ are *almost disjoint or a.d.* if $a \cap b$ is finite.
- We say that a family $\mathscr{A} \subset [\omega]^{\omega}$ is *almost disjoint or a.d.* if its members are pairwise almost disjoint.
- A *Maximal Almost Disjoint family, or MAD family* is an infinite a.d. family that is not properly contained in a larger a.d. family.
- Equivalently, an infinite a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ is MAD iff $\forall b \in [\omega]^{\omega} \exists a \in \mathscr{A} \, [|b \cap a| = \omega]$.

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- For example the size of $\mathscr{A}$.
- If we want to make $|\mathscr{A}|$ as large as possible, then we can, but we need an intermediate step.
- Identify $\omega$ with $2^{<\omega}$. Then the branches form an a.d. family of size $\mathfrak{c}$. Extend it to a MAD family.

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## Definition

$\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A} \subset [\omega]^{\omega}$ *and* $\mathscr{A}$ *is a MAD family*$\}$.

- The value of $\mathfrak{a}$ is not decided in $\mathrm{ZFC}$.
- There are several such *cardinal invariants*.
- Play a crucial role in many combinatorial constructions.
- Usually take the form of the least size of a family of a certain sort.

# Definitions and motivations

- for $a, b \in \mathcal{P}(\omega)$, $a$ *splits* $b$ if $|a \cap b| = |(\omega \setminus a) \cap b| = \omega$.
- $F \subset \mathcal{P}(\omega)$ is called a *splitting family* if $\forall b \in [\omega]^{\omega} \exists a \in F\, [a$ splits $b]$.

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- A family $F \subset \omega^{\omega}$ is called *unbounded* if it has no upper bound in $\langle \omega^{\omega}, \leq^{*} \rangle$.
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## Definition

$\mathfrak{b} = \min\{|F| : F \subset \omega^{\omega}$ *is an unbounded family*$\}$.
$\mathfrak{d} = \min\{|F| : F \subset \omega^{\omega}$ *is a dominating family*$\}$.

# Definitions and motivations

## Definition

*For any family $\mathscr{A} \subset \mathcal{P}(\omega)$, the ideal generated by $\mathscr{A}$ (together with the Fréchet ideal) is denoted by $\mathcal{I}(\mathscr{A})$.*

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*For any ideal $\mathcal{I}$ on $\omega$, $\mathcal{I}^+$ denotes $\mathcal{P}(\omega) \setminus \mathcal{I}$. The sets in $\mathcal{I}^+$ are called $\mathcal{I}$-positive. $\mathcal{I}^*$ denotes $\{\omega \setminus a : a \in \mathcal{I}\}$, this is the dual filter to $\mathcal{I}$. An ideal $\mathcal{I}$ is said to be tall if $\forall b \in [\omega]^{\omega} \exists a \in [b]^{\omega} [a \in \mathcal{I}]$.*

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- We are interested in almost disjoint families for which $\mathcal{I}(\mathscr{A})$ enjoys certain strong properties.

# Definitions and motivations

- If $\mathscr{A}$ is a.d., then $\mathcal{I}^+(\mathscr{A})$ always has a strong combinatorial property.

**Theorem**

*If $\mathscr{A} \subset \mathcal{P}(\omega)$ is an infinite a.d. family, then $\mathcal{I}^+(\mathscr{A})$ is a selective co-ideal.*

# Definitions and motivations

- If $\mathscr{A}$ is a.d., then $\mathcal{I}^+(\mathscr{A})$ always has a strong combinatorial property.

**Theorem**

*If $\mathscr{A} \subset \mathcal{P}(\omega)$ is an infinite a.d. family, then $\mathcal{I}^+(\mathscr{A})$ is a selective co-ideal.*

- This essentially means that $\mathcal{I}^*(\mathscr{A})$ "can be" extended to a Ramsey ultrafilter.

**Definition**

*$\mathcal{I}^+$ is called a selective coideal if for every sequence $e_0 \supset e_1 \supset \cdots$, with $e_i \in \mathcal{I}^+$, there is an $e = \{n_0 < n_1 < \cdots\} \in \mathcal{I}^+$ such that $n_0 \in e_0$ and $n_{i+1} \in e_{n_i}$ for each $i$.*

# Definitions and motivations

- The main point is the following:

**Lemma**

*Suppose $\mathscr{A}$ is an a.d. family. Suppose $b \subset \omega$ and $\exists^{\infty} a \in \mathscr{A}\ [|a \cap b| = \omega]$. Then $b \in \mathcal{I}^{+}(\mathscr{A})$*

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- The main point is the following:

**Lemma**

*Suppose* $\mathscr{A}$ *is an a.d. family. Suppose* $b \subset \omega$ *and* $\exists^{\infty} a \in \mathscr{A}\ [|a \cap b| = \omega]$*. Then* $b \in \mathcal{I}^{+}(\mathscr{A})$

**Proof.**

If $b \in \mathcal{I}(\mathscr{A})$, then there exist $a_0, \ldots a_k \in \mathscr{A}$ such that $b \subset^{*} a_0 \cup \cdots \cup a_k$. By hypothesis, there is $a \in \mathscr{A} \setminus \{a_0, \ldots, a_k\}$ such that $a \cap b$ is infinite. However $a \cap b$ is a.d. from $a_0 \cup \cdots \cup a_k$ and yet $a \cap b \subset b \subset^{*} a_0 \cup \cdots \cup a_k$. This is a contradiction. ⊣

# Definitions and motivations

- We are interested in families where there is a strong combinatorial relationship between $\mathscr{A}$ and $\mathcal{I}^+(\mathscr{A})$.
- A typical example is the following:

**Definition**

*An almost disjoint family $\mathscr{A}$ is tight (also called $\aleph_0$-MAD) if for any $\{b_n : n \in \omega\} \subset \mathcal{I}^+(\mathscr{A})$, there is $a \in \mathscr{A}$ such that $\forall n \in \omega \, [|a \cap b_n| = \aleph_0]$.*

- This asks for a $\sigma$-version of maximality.
- It is also connected with the notion of indestructible MAD families.

# Definitions and motivations

## Definition

*Let $\mathbb{P}$ be a notion of forcing. A MAD family $\mathscr{A} \subset [\omega]^{\omega}$ is called $\mathbb{P}$-indestructible if $\Vdash_{\mathbb{P}} \mathscr{A}$ is MAD.*

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## Definition

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- Obviously, if $\mathbb{P}$ does not add reals, then every MAD $\mathscr{A}$ is $\mathbb{P}$-indestructible.
- If a MAD $\mathscr{A} \subset [\omega]^{\omega}$ is indestructible for any $\mathbb{P}$ that adds a real, then $\mathscr{A}$ is also Sacks indestructible.

## Theorem

*Every tight a.d. family is Cohen-indestructible. If a MAD family $\mathscr{A}$ is Cohen-indestructible, then for some $X \in \mathcal{I}^{+}(A)$, $\mathscr{A} \upharpoonright X = \{X \cap a : a \in \mathscr{A}\}$ is tight.*

# Definitions and motivations

## Definition

*An a. d. family $\mathscr{A}$ is called weakly tight if for all $\{b_n : n \in \omega\} \subset \mathcal{I}^+(\mathscr{A})$, there is $a \in \mathscr{A}$ such that $\exists^\infty n \in \omega\, [|a \cap b_n| = \aleph_0]$.*

- This is a natural weakening of *tight* investigated by [1].
- It is connected to the Katetov order on a.d. families.

# Definitions and motivations

## Definition

*An a.d. family $\mathscr{A}$ is called Laflamme if $\mathscr{A}$ is not contained in any $F_\sigma$ ideal on $\omega$.*

Considered by Laflamme in 1992 [2] (in connection with destroying MAD families without adding unbounded reals).

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Considered by Laflamme in 1992 [2] (in connection with destroying MAD families without adding unbounded reals).

## Theorem

*If $\mathcal{I}$ is any $F_\sigma$ ideal on $\omega$, then there is a proper $\omega^\omega$-bounding forcing $\mathbb{P}_{\mathcal{I}}$ which adds an element of $[\omega]^\omega$ that is almost disjoint from every element of $\mathbf{V} \cap \mathcal{I}$.*

# Definitions and motivations

- Laflamme's questions is related to the problem of whether $\mathfrak{d} = \aleph_1$ implies $\mathfrak{a} = \aleph_1$.
- If you can get all MAD families to be contained in $F_\sigma$ ideals, then you could hope to increase $\mathfrak{a}$ without increasing $\mathfrak{d}$.
- We will see that when $\mathfrak{d} = \aleph_1$, Laflamme families exist.

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*An a. d. family is called completely separable if* $\forall b \in \mathcal{I}^{+}(\mathscr{A}) \exists a \in \mathscr{A}\ [a \subset b]$.

- This question has a long history. It is connected with the existence of ADRs.

## Definition

*Given* $\mathscr{C} \subset [\omega]^{\omega}$, *we say that a family* $\mathscr{A} = \{a_c : c \in \mathscr{C}\} \subset [\omega]^{\omega}$ *is an almost disjoint refinement (ADR) of* $\mathscr{C}$ *if*

1. $\forall c \in \mathscr{C}\ [a_c \subset c]$
2. $\forall c_0, c_1 \in \mathscr{C} \left[c_0 \neq c_1 \implies \left|a_{c_0} \cap a_{c_1}\right| < \omega\right]$.

# Definitions and motivations

## Fact

*Some facts:*

- *If* $\mathscr{C} \subset [\omega]^{\omega}$ *has an ADR, then there is tall ideal* $\mathcal{I}$ *such that* $\mathcal{I} \cap \mathscr{C} = 0$.
- $\mathcal{I}^{+}$ *has an ADR for every tall* $\mathcal{I}$ *iff for every tall* $\mathcal{I}$ *there is a completely separable* $\mathscr{A} \subset \mathcal{I}$.
- *If* $\mathscr{A}$ *is completely separable, then for every* $b \in \mathcal{I}^{+}(\mathscr{A})$, *there are* $\mathfrak{c}$ *many* $a \in \mathscr{A}$ *such that* $a \subset b$.

# Definitions and motivations

**Basic Question**

*When do these a. d. families exist? Do any of them exist in* $\mathrm{ZFC}$*?*

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*When do these a. d. families exist? Do any of them exist in* $\mathrm{ZFC}$*?*

- They all exist under $\mathrm{CH}$.
- In these talks we will first survey some of the recent progress on proving existence.
- Then we focus on completely separable and on weakly tight families.
- Both types of families exist if $\mathfrak{c} < \aleph_\omega$ (full proofs, time permitting).

# Recent progress

**Theorem (Shelah[3], 2010)**

*If $\mathfrak{c} < \aleph_\omega$, then there is a completely separable a. d. family.*

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- The proof is in 3 cases:
  1. $\mathfrak{s} < \mathfrak{a}$
  2. $\mathfrak{s} = \mathfrak{a}$ + a certain PCF-type assumption holds.
  3. $\mathfrak{a} < \mathfrak{s}$ + a different PCF-type assumption holds.

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- The proof is in 3 cases:
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  2. $\mathfrak{s} = \mathfrak{a}$ + a certain $\mathrm{PCF}$-type assumption holds.
  3. $\mathfrak{a} < \mathfrak{s}$ + a different $\mathrm{PCF}$-type assumption holds.
- The $\mathrm{PCF}$ type assumptions both automatically hold if $\mathfrak{c} < \aleph_\omega$.
- This proof is the basis for all the recent progress.

# Recent progress

- The $\mathrm{PCF}$ assumption can be eliminated from case 2 of Shelah's construction.

**Theorem (Mildenberger, R., and Steprans)**

*If $\mathfrak{s} \leq \mathfrak{a}$, then there is a completely separable MAD family.*

- The main point in this proof is that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.

# Recent progress

**Theorem (R. and Steprans)**

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I recently improved this to

**Theorem (R.)**

*If $\mathfrak{c} < \aleph_\omega$, then there is a weakly tight family.*

- The proof is broken down into 2 analogous cases:
  1. $\mathfrak{s} \leq \mathfrak{b}$
  2. $\mathfrak{b} < \mathfrak{s}$ + a certain PCF type assumption.
- Again the $\mathrm{PCF}$ type assumption is automatically satisfied if $\mathfrak{c} < \aleph_\omega$.

# Recent progress

- Let us say that a family $\mathcal{F} \subset \mathcal{P}(\omega)$ is $F_\sigma$ *splitting* if for each $F_\sigma$ ideal $\mathcal{I}$ on $\omega$, there exists $a \in \mathcal{F}$ such that both $a$ and $\omega \setminus a$ are in $\mathcal{I}^+$.

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$\mathfrak{p}(\mathcal{F}_\sigma) = \min\{\mathfrak{p}(\mathcal{F}) : \mathcal{F}$ *is a tall* $F_\sigma$ *– filter*$\}$.

# Recent progress

- $\mathfrak{p}(\mathcal{F}_\sigma)$ is consistently bigger than $\mathfrak{d}$.
- $\mathrm{add}(\mathcal{N}) \leq \mathfrak{p}(\mathcal{F}_\sigma)$
- $\mathfrak{s}(\mathcal{F}_\sigma) \leq \min\{\max\{\mathfrak{b}, \mathfrak{s}\}, \mathrm{non}(\mathcal{N})\}$.

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2. *If $\mathfrak{b} \leq \mathfrak{p}(\mathcal{F}_\sigma) < \aleph_\omega$, then there is a Laflamme family.*

- There are 2 cases:
  1. $\mathfrak{s}(\mathcal{F}_\sigma) \leq \mathfrak{p}(\mathcal{F}_\sigma)$.
  2. $\mathfrak{b} \leq \mathfrak{p}(\mathcal{F}_\sigma)$ + a PCF-type assumption.

# Recent progress

## Corollary

1. *If* $\mathfrak{b} = \mathfrak{s} = \aleph_1$*, then there is a Laflamme family.*
2. *If* $\text{non}(\mathcal{N}) = \aleph_1$*, then there is a Laflamme family.*

# Questions

## Question

*Is there a Laflamme family assuming* $\mathfrak{c} < \aleph_\omega$*?*

- What is still open is the case: $\mathfrak{p}(\mathcal{F}_\sigma) < \min\{b, \mathfrak{s}(\mathcal{F}_\sigma)\}$.
- An interesting sub-question is what happens when $\mathfrak{b} = \mathfrak{c}$?

# Questions

## Question

*Is there a Sacks indestructible MAD family assuming $\mathfrak{c} < \aleph_\omega$?*

- A MAD family $\mathscr{A} \subset [\omega]^\omega$ is Sacks indestructible iff for each 1-1 map $\Sigma : 2^{<\omega} \to \omega$, there exists $a \in \mathscr{A}$ such that $\exists^{\mathfrak{c}} f \in 2^\omega \left[ |a \cap (\Sigma''\{f \restriction n : n \in \omega\})| = \omega \right]$.
- If $\mathfrak{a} < \mathfrak{c}$, then any MAD family of size $\mathfrak{a}$ is Sacks indestructible. So you can assume $\mathfrak{a} = \mathfrak{c}$ for free.

# Questions

## Question

*Can the general method be modified to construct MAD families in $\omega^\omega$ with special properties?*

# Bibliography

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