Almost disjoint refinements

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Hejnice 2014

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Motivation

Our senior hosts and I like this topic...

One more thing before refinements. . .

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The concept

Given a family

$$\mathcal{H} = \{H_0, H_1, H_2, \ldots, H_\alpha, \ldots\} \text{ of “large” sets}$$

we (at least some of us) want to find a family

$$\begin{matrix} \cup\!| & \cup\!| & \cup\!| & \ldots & \cup\!| & \ldots \end{matrix}$$

$$\mathcal{A} = \{A_0, A_1, A_2, \ldots, A_\alpha, \ldots\} \text{ of still “large” sets s.t.}$$

$$A_\alpha \cap A_\beta \text{ is “small” for every } \alpha \neq \beta.$$

“large” and “small”

Definition

Given an ideal $\mathcal{I}$ on a set X. A family $\mathcal{A} \subseteq \mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ is ***$\mathcal{I}$-almost disjoint*** ($\mathcal{I}$-AD) if $A \cap B \in \mathcal{I}$ for every two distinct $A, B \in \mathcal{A}$.

From now on “large” $=$ “$\in \mathcal{I}^+$”, “small” $=$ “$\in \mathcal{I}$”, and we are looking for $\mathcal{I}$-AD refinements of families of $\mathcal{I}$-positive sets.
For instance, $\mathcal{I}^+$ does not have $\mathcal{I}$-AD refinements.

Analytic and coanalytic ideals

Proposition

If $\mathcal{I}$ is an analytic or coanalytic ideal on ω, then there are perfect $\mathcal{I}$-AD families on every $X \in \mathcal{I}^+$.

Proof: It is enough working on ω because $\mathcal{I} \restriction X$ is a continuous preimage of $\mathcal{I}$ (hence also (co)analytic).

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$\mathcal{I}$ is meager hence (by Talagrand's characterization) there is a partition $(P_n)_{n\in\omega}$ of ω into finite sets s.t. $|\{n \in \omega : P_n \subseteq A\}| < \omega$ for every $A \in \mathcal{I}$.

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Consider the family $\mathcal{A}_0$ of branches of $2^{<\omega}$ in $\mathcal{P}(2^{<\omega})$. It is a perfect (Fin-)AD family.

For each $A \in \mathcal{A}_0$ let $A' = \bigcup\{P_n : n \in A\}$, $A' \in \mathcal{I}^+$, and let $\mathcal{A} = \{A' : A \in \mathcal{A}_0\}$. The function $\mathcal{P}(\omega) \to \mathcal{P}(\omega)$, $A \mapsto A'$ is injective and continuous hence $\mathcal{A}$ is also perfect.

Generalizations to higher complexity?

Remark

In L there is a Δ^1_2 ideal $\mathcal{I}$ such that all $\mathcal{I}$-AD families are countable:

We know that there is a Δ^1_2 prime ideal $\mathcal{J}$ in L (by the most natural recursive construction of a prime ideal from a Δ^1_2-good well-order of the reals).

Copy $\mathcal{J}$ to every elements of an infinite partition of ω, and let $\mathcal{I}$ be the generated ideal.

Refinements of $\mathcal{I} \cap V$ in $V^{\mathbb{P}}$

Theorem (Brendle, Farkas, Khomskii)

Let $\mathcal{I}$ be an analytic or coanalytic ideal and assume that $\mathbb{P}$ adds new reals. Then

$$V^{\mathbb{P}} \models \text{“}\mathcal{I}^+ \cap V \text{ has an } \mathcal{I}\text{-AD refinement.”}$$

In other words, there is a family $\{A_X : X \in \mathcal{I}^+ \cap V\}$ in $V^{\mathbb{P}}$ s.t. (i) $A_X \subseteq X$, $A_X \in \mathcal{I}^+$ for every $X \in \mathcal{I}^+ \cap V$ and (ii) if $X \neq Y$ then $A_X \cap A_Y \in \mathcal{I}$.

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Proof: In a minute but first of all some examples. . .

Examples of Borel and (co)analytic ideals

F_σ ideals:

- Summable ideals, e.g. $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n\in A} \frac{1}{n} < \infty\right\}$.
- Farah's ideal:
  $\mathcal{J}_F = \left\{A \subseteq \omega : \sum_{n\in\omega} \frac{\min\{n, |A\cap[2^n, 2^{n+1})|\}}{n^2} < \infty\right\}$.
- Tsirelson ideals (Farah, Solecki, Veličković).
- The van der Waerden ideal:
  $\mathcal{W} = \{A \subseteq \omega : A \text{ does not contain arbitrary long AP's}\}$.
- The random graph ideal:
  $\mathrm{Ran} = \mathrm{id}(\{\text{homogeneous subsets of the random graph}\})$.
- The ideal of graphs with finite chromatic number:
  $\mathcal{G}_{\mathrm{fc}} = \{E \subseteq [\omega]^2 : \chi(\omega, E) < \omega\}$.

Examples of Borel and (co)analytic ideals

$F_{\sigma\delta}$ ideals:

- (Generalized) Density ideals, e.g. $\mathcal{Z} = \left\{A \subseteq \omega : \frac{|A \cap n|}{n} \to 0\right\}$.
- The trace of the null ideal: $\mathrm{tr}(\mathcal{N}) = \left\{A \subseteq 2^{<\omega} : \lambda\{f \in 2^{\omega} : \exists^{\infty} n\ f \restriction n \in A\} = 0\right\}$.
- The ideal of nowhere dense subsets of the rationals: $\mathrm{Nwd} = \left\{A \subseteq \mathbb{Q} : \mathrm{int}(\overline{A}) = \emptyset\right\}$.
- Banach space ideals (Louveau, Veličković).

$F_{\sigma\delta\sigma}$ ideals:

- The ideal Conv is generated by those infinite subsets of $\mathbb{Q} \cap [0,1]$ which are convergent in $[0,1]$, in other words $\mathrm{Conv} = \left\{A \subseteq \mathbb{Q} \cap [0,1] : |\{\text{acc. points of } A \text{ (in } \mathbb{R})\}| < \omega\right\}$.
- The Fubini product of Fin by itself: $\mathrm{Fin} \otimes \mathrm{Fin} = \left\{A \subseteq \omega \times \omega : \forall^{\infty}\ n \in \omega\ |(A)_n| < \omega\right\}$.

Examples of Borel and (co)analytic ideals

A coanalytic(-complete) example, the ideal of graphs without infinite complete subgraphs:

$$\mathcal{G}_{\mathrm{c}} = \{E \subseteq [\omega]^2 : \forall\, X \in [\omega]^\omega\ [X]^2 \nsubseteq E\}.$$

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Proof: Fix perfect $\mathcal{I}$-AD families $\mathcal{A}_X$ on every $X \in \mathcal{I}^+$. The statement “$\mathcal{A}_X \subseteq \mathcal{I}^+$ and $\mathcal{A}_X$ is $\mathcal{I}$-AD” is $\undertilde{\Pi}^1_2$ hence absolute.

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For every $X, Y \in \mathcal{I}^+$ let $B(X, Y) = \{A \in \mathcal{A}_X : A \cap Y \in \mathcal{I}^+\}$. Then $B(X, Y)$ is a continuous preimage of $\mathcal{I}^+$ (under $\mathcal{A}_X \to \mathcal{P}(\omega)$, $A \mapsto A \cap Y$), hence it is also (co)analytic.

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Working in $V^{\mathbb{P}}$, enumerate $\{X_\alpha : \alpha < \mathfrak{c}^V\}$ the set $\mathcal{I}^+ \cap V$. We will construct the desired $\mathcal{I}$-AD refinement $\{A_\alpha : \alpha < \mathfrak{c}^V\}$ by recursion on $\mathfrak{c}^V$. We will also define a sequence $(B_\alpha)_{\alpha < \mathfrak{c}^V}$ in $\mathcal{I}^+$.

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$B(X, Y) = \{A \in \mathcal{A}_X : A \cap Y \in \mathcal{I}^+\}$.
$\{X_\alpha : \alpha < \mathfrak{c}^V\} = \mathcal{I}^+ \cap V$.
We construct the $\mathcal{I}$-AD refinement $\{A_\alpha : \alpha < \mathfrak{c}^V\}$ and the sequence $(B_\alpha)_{\alpha < \mathfrak{c}^V}$ in $\mathcal{I}^+$.
Assume that $\{A_\xi : \xi < \alpha\}$ and $(B_\xi)_{\xi < \alpha}$ are done, and let

$$\gamma_\alpha = \min\{\gamma : B(X_\gamma, X_\alpha) \text{ contains a perfect set (in } V)\} \leq \alpha.$$

Let $B_\alpha \in B(X_{\gamma_\alpha}, X_\alpha) \setminus (V \cup \{B_\xi : \xi < \alpha\})$ and $A_\alpha = X_\alpha \cap B_\alpha \in \mathcal{I}^+$.

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Let $B_\alpha \in B(X_{\gamma_\alpha}, X_\alpha) \setminus (V \cup \{B_\xi : \xi < \alpha\})$ and $A_\alpha = X_\alpha \cap B_\alpha \in \mathcal{I}^+$.
We claim that $\{A_\alpha : \alpha < \mathfrak{c}^V\}$ is an $\mathcal{I}$-AD family. Let $\alpha \neq \beta$.
Case 1: If $\gamma_\alpha = \gamma_\beta = \gamma$ then $B_\alpha, B_\beta \in \mathcal{A}_{X_\gamma}$ are distinct, and hence $A_\alpha \cap A_\beta \subseteq B_\alpha \cap B_\beta \in \mathcal{I}$.

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Case 2: $\gamma_\alpha < \gamma_\beta$. Then $B(X_{\gamma_\alpha}, X_\beta)$ does not contain perfect subsets.

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Case 2: $\gamma_\alpha < \gamma_\beta$. Then $B(X_{\gamma_\alpha}, X_\beta)$ does not contain perfect subsets.

Case 2a: If $\mathcal{I}$ is coanalytic, then $B(X_{\gamma_\alpha}, X_\beta)$ is analytic hence countable in V and so it is the same set in $V^{\mathbb{P}}$, in particular $B_\alpha \notin B(X_{\gamma_\alpha}, X_\beta)$, hence $A_\alpha \cap A_\beta \subseteq B_\alpha \cap X_\beta \in \mathcal{I}$.

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Case 2b: If $\mathcal{I}$ is analytic, then $B(X_{\gamma_\alpha}, X_\beta)$ is coanalytic. Therefore if $B(X_{\gamma_\alpha}, X_\beta)$ does not contain perfect subsets from V, then it is the same set in the extension: If it would contain a new real $E \in B(X_{\gamma_\alpha}, X_\beta) \setminus V$, then $E \in B(X_{\gamma_\alpha} X_\beta) \setminus L[r]$ but it contradicts the Mansfield-Solovay theorem. In particular, $B_\alpha \notin B(X_{\gamma_\alpha}, X_\beta)$ and it yields that $A_\alpha \cap A_\beta \subseteq B_\alpha \cap X_\beta \in \mathcal{I}$.

Questions

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Does there exist (consistently) a $\underset{\sim}{\Sigma}^1_2$ or $\underset{\sim}{\Pi}^1_2$ ideal $\mathcal{I}$ and a forcing notion $\mathbb{P}$ which adds new reals such that the definition of $\mathcal{I}$ is absolute between V and $V^{\mathbb{P}}$ but $V^{\mathbb{P}} \models$"$\mathcal{I}^+ \cap V$ has no $\mathcal{I}$-AD refinements"?

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Question

Is it possible that $\mathbb{P}$ adds new reals, does not collapse cardinals and $V^{\mathbb{P}} \models$"$[\omega]^\omega \cap V$ has a projective AD refinement"?

Thank you for your attention!