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On σ-fields invariant under operation (A)

This note is a summary of my lecture presented at the Nineteenth Summer Symposium in Real Analysis on June 1995. Through the note, $\mathcal{A}$ always denotes a σ-field of subsets of a set X and $\mathcal{N}(\mathcal{A}) = \{A : \mathcal{P}(A) \subseteq \mathcal{A}\}$. It is said that $\mathcal{A}$ has *hull property* if for every $Y \subseteq X$ there is $A \in \mathcal{A}$ such that $Y \subseteq A$ and for every $B \subseteq A \setminus Y$, $B \in \mathcal{N}(\mathcal{A})$ [1]. A function m_e is called a *quasi-outer measure* if $m_e : X \to [0, \infty]$, $m(\emptyset) = 0$ and $m(C) \leq m(A) + m(B)$ for every $A, B, C \subseteq X$ such that $C \subseteq A \cup B$. A quasi-outer measure which is countably subadditive is (of course) called an outer measure (in the sense of Carathéodory). Let

$$\mathcal{M}(m_e) = \{A \subseteq X : m_e(Y) = m_e(Y \cap A) + m_e(Y \setminus A) \quad \text{for every} Y \subseteq X\}.$$

A function $m : \mathcal{A} \to [0, \infty]$ is called a *quasi-measure* if $m(\emptyset) = 0$ and m is finitely additive. If m is additionally countable additive then m is called a *measure.* A quasi measure m is called *complete* if $\{A \in \mathcal{A} : m(A) = 0\} \subseteq \mathcal{N}(\mathcal{A})$.

We investigate relations between the following seven conditions.

(C_1) $\mathcal{A}$ is closed under operation (A) of Suslin (by this operation I understand that well known one which when applied to the σ-field of Borel sets gives the family of analitic sets).

(C_2) $\mathcal{A}$ has hull property.

(C_3) $\mathcal{A} \setminus \mathcal{N}(\mathcal{A})$ satisfies countable chain conditions.

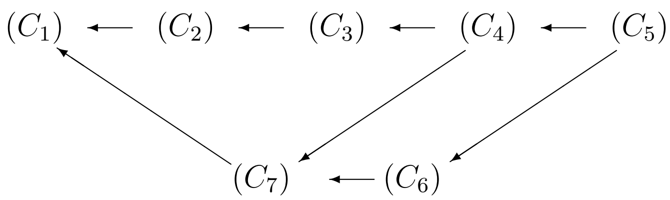
(C_4) There exists a finite complete quasi measure on $\mathcal{A}$.

(C_5) There exists a finite complete measure on $\mathcal{A}$.

(C_6) There exists an outer measure μ_e on $\mathcal{A}$ such that $\mathcal{A} = \mathcal{M}(\mu_e)$.

(C_7) There exists an outer quasi measure m_e on $\mathcal{A}$ such that $\mathcal{A} = \mathcal{M}(m_e)$

We know about the following implications:



The most important are $(C_2) \to (C_1)$ (Theorem of Marczewski from [1]) and $(C_6) \to (C_1)$ (Theorem of Saks from [2]).

To prove $(C_7) \to (C_1)$ we modify a proof of C.Ryll- Nardzewski of the mentioned theorem of Saks. We also know that $(C_2) \not\to (C_3)$, $(C_1) \not\to (C_7)$, $(C_6) \not\to (C_2)$ and $(C_4) \not\to (C_6)$. To prove $(C_4) \not\to (C_6)$ we use a theorem of Tarski about invariant measures [4].

The following question is open. Does $(C_3) \not\to (C_7)$?

If **no** then there are no other open questions about our diagram. Additionally, an example that $(C_3) \not\to (C_7)$ would easily give an example of a C.C.C. Boolean algebra without strictly positive finite quasi measure (the existence of which was discovered by Gaifman [3]).

References

[1] E.Szpilrajn-Marczewski, Sur certains invariants de l'operation (A), Fund.Math.21 (1933), 229-235.

[2] S.Saks, Theory of the integral, Warszawa- Lwów 1937.

[3] H.Gaifman, Concerning measures on Boolean algebras, Pacific.J.Math. 14 (1964), 61-73.

[4] A.Tarski, Über das absolute Mass linear Punktmengen, Fund.Math. 30 (1938), 218-234.