Countable dense homogeneous filters and the covering property of Rothberger

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Basic definitions and problems

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The following question has been stated is recent papers of A. Medini, D. Milovich and R. Hernández-Gutiérrez, M. Hrušák.

Question

Characterize non-meager filters on ω which are CDH when considered with the topology inherited from 2^ω. □

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Every non-meager P filter is CDH. □

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Natural attempt because there are ZFC examples of non-meager P^- filters, both being P^- filters and CDH filters are weakenings of being a P-filter.

CDH filters and Rothberger's property

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Rothberger property:

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The Rothberger covering property of a base of a filter $\mathcal{U}$ implies the following combinatorial property of $\mathcal{U}$:

> *For each sequence $\langle U_m : m < \omega \rangle \in \mathcal{U}^+$ there is a sequence $\langle k_m : m < \omega \rangle$ with each $k_m \in U_m$ and $\{k_m : m \in \omega\} \in \mathcal{U}^+$.*

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Indeed, every $X \in \mathcal{U}^+$ in a natural way gives rise to an open cover $\mathcal{O}_X$ of $\mathcal{U}$.

Theorem

It is consistent that there exists a non-meager non-CDH filter $\mathcal{U}$ on ω which has a base with the Rothberger covering property. □

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Indeed, every $X \in \mathcal{U}^+$ in a natural way gives rise to an open cover $\mathcal{O}_X$ of $\mathcal{U}$. So our attempt to construct a ZFC example of a CDH filter has no chance.

Ideas of the proof

Lemma

Assume that CH holds in V. Then in $V^{\mathbb{C}_{\omega_1}}$ there exists a centered family $\mathcal{A} \subset V \cap [\omega]^\omega$ such that no filter $\mathcal{U} \supset \mathcal{A}$ with a base consisting of ground model reals is CDH.

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Lemma (M. Scheepers, F. Tall 2010)

In $V^{\mathbb{C}_{\omega_1}}$ every subspace of $V \cap [\omega]^\omega$ has the Rothberger covering property.

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Proof of Theorem.

Throughout the proof we shall work in $V[H]$, where H is a $\mathbb{C}_{\omega_1}$-generic filter over a model V of CH. Let $\mathcal{A} \in V[H]$, $\mathcal{A} \subset [\omega]^\omega \cap V$ be such as above.

Proof of the theorem, continued

Let $\mathcal{F}$ be any ultrafilter containing $\mathcal{A}$ and $\mathcal{U}$ be the filter generated by $\mathcal{F} \cap V$.

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$\mathcal{U}$ is non-meager because $\mathcal{U} \cap V$ is a "half" of $V \cap 2^\omega$ and old reals remain non-meager after adding Cohen reals.

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$\mathcal{U}$ has a Rothberger base, namely $\mathcal{U} \cap V$. □

The last slide

Thank you for your attention.