Some remarks on splittings

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Splittings

Let $\mathcal{A}$ be a collection of pairwise disjoint families of ω.
For $x \subseteq \omega$ denote $x^0 = x$ and $x^1 = \omega \setminus x$.
The following is taken from A. Kamburelis and B. Węglorz, *Splittings*, Arch. Math. Logic 35 (1996).

Splitting family

We say that $s \in [\omega]^\omega$ *splits* a disjoint family $\{a_n\} \in \mathcal{A}$ iff

$$\forall_{i<2} \exists_n^\infty a_n \subseteq s^i$$

and $\mathcal{B} \subseteq [\omega]^\omega$ is called *a splitting family w.r.t.* $\mathcal{A}$ if any $A \in \mathcal{A}$ is splitted by some member of $\mathcal{B}$.

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Splitting numbers

Define *the splitting number w.r.t.* $\mathcal{A}$ as

$$s(\mathcal{A}) = \min \{|\mathcal{B}| : \mathcal{B} \text{ is a splitting family w.r.t. } \mathcal{A}\}.$$

Splittings

If $[\{\{n\} : n < \omega\}]^\omega \subseteq \mathcal{A}_0 \subseteq \mathcal{A}_1$, then $\mathfrak{s} \leqslant \mathfrak{s}(\mathcal{A}_0) \leqslant \mathfrak{s}(\mathcal{A}_1)$.

We say that $\mathcal{B}$ is a *block-splitting* family if it is $\mathcal{A}$-splitting for $\mathcal{A}$ a collection of infinite families of pairwise disjoint finite subsets of ω.

We say that $\mathcal{B}$ is a *weakly* ω*-splitting* (in short (ω, ω)*-splitting*) family if it is $\mathcal{A}$-splitting for $\mathcal{A}$ a collection of infinite pairwise disjoint subfamilies of $[\omega]^\omega$.

The corresponding splitting numbers are denoted by $\mathfrak{s}_{\text{BLOCK}}$ and $\mathfrak{s}_{\omega,\omega}$.

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Some facts

1. $s_{\text{BLOCK}} = \max\{b, s\}$ (A.Kamburelis, B.Węglorz (1996))
2. $s_{\omega,\omega} = s$ (H. Mildenberger, D.Raghavan, J.Steprāns (2012)) and if $b \leqslant s$ then any block-splitting family is (ω, ω)-splitting.

Motivation

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Erdös-Shelah Problem (1972)

ZFC $\vdash$ *there exists completely separable MAD families* ???

S.Shelah, *MAD saturated families and SANE Player*, Can. J. Math. 63(2011)

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One can remove pcf-like assumptions?

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One can remove pcf-like assumptions?

For the second case - YES - as proven by H. Mildenberger, D.Raghavan, J.Steprāns in *Splitting families and complete separability* (2012, to appear in Can. Bull. Math.).

Completely separable MAD family from $\mathfrak{s} \leqslant \mathfrak{a}$

Sketch of the (Shelah)-Mildenberger-Raghavan-Steprāns proof of the existence completely separable MAD family from $s \leqslant a$. Using a witness $\{x_\xi \in [\omega]^\omega : \xi < s\}$ for $s_{\omega,\omega} = s$ build $\mathcal{A}$ by an induction of the length $\mathfrak{c}$ by extending at each stage $\delta < \mathfrak{c}$ a partial family $\mathcal{A}_\delta = \mathcal{A} \upharpoonright \delta = \{a_{\sigma_\alpha} : \alpha < \delta\}$ indexed by nodes of the tree $2^{<s}$.

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Splitting Lemma

If $b \in \mathcal{I}^+(\mathcal{A}_\delta)$ then one can find x_α, $\alpha < s$, splitting b into $\mathcal{I}(\mathcal{A}_\delta)$-positive pieces, i.e. for both $i < 2$ it holds $b \cap x_\alpha^i \in \mathcal{I}^+(\mathcal{A}_\delta)$.

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For $\eta \in 2^{<s}$ define a family of pseudointersections as follows

$$\mathcal{I}_\eta = \left\{ a \in [\omega]^\omega : \forall_{\xi < \mathrm{dom}(\eta)}\ a \subseteq_* x_\xi^{\eta(\xi)} \right\}.$$

Completely separable MAD family from $\mathfrak{s} \leqslant \mathfrak{a}$

Main Lemma

Let $\mathfrak{s} \leqslant \mathfrak{a}$ and $\delta < \mathfrak{c}$. For any $b \in \mathcal{I}^+(\mathcal{A}_\delta)$ one can find $\sigma \in 2^{<\mathfrak{s}}$ such that $\sigma \subsetneq \sigma_\alpha$ for all $\alpha < \delta$ and $a \in \mathcal{I}_\sigma \cap [b]^\omega$ such that $\mathcal{A}_\delta \cup \{a\}$ is almost disjoint family.

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Sketch/ideas/picture of the proof

Use *Splitting Lemma* and x_ξ's to construct a perfect subtree of $\{\sigma_s : s \in 2^{<\omega}\}$ of $2^{<s}$ and $\{b_s : s \in 2^{<\omega}\} \subseteq \mathcal{I}^+(\mathcal{A}_\delta)$ such that for all $s \in 2^{<\omega}$, $i < 2$ and $\gamma < dom(\sigma_s)$ it holds

- $b_s \cap x_\gamma^{1-\sigma_s(dom(\sigma_s))} \in \mathcal{I}(\mathcal{A}_\delta)$ and $b_{s\hat{\ }i} = b_s \cap x^i_{dom(\sigma_s)}$,
- $b_0 = b$ and $b_s \cap x^i_{dom(\sigma_s)} \in \mathcal{I}^+(\mathcal{A}_\delta)$.

Completely separable MAD family from $s \leqslant a$

Sketch/ideas/picture of the proof - continued

Choose some branch f satisfying $\tau_f \subsetneq \sigma_\alpha$ for all $\alpha < \delta$ (here, τ_f is a supremum of nodes from $2^{<s}$ indexed by the branch f). One can ensure that a "comb" between consecutive nodes of the branch consists of $\mathcal{I}(\mathcal{A}_\delta)$-small members and there exists $a' \in \mathcal{I}_{\tau_f}$ (vide blackboard). In other words

$$(\forall \xi < dom(\tau_f))(\exists F_\xi \in [\delta]^{<\omega}) \left[a' \cap x_\xi^{1-\tau_f(\xi)} \subseteq_* \bigcup \{a_\alpha : \alpha \in F_\xi\} \right].$$

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Putting $\mathcal{F} = \bigcup\{F_\xi : \xi < \tau_f\}$ and $\mathcal{G} = \{\alpha < \delta : \sigma_\alpha \subseteq \tau_f\}$, we see $|\mathcal{F} \cup \mathcal{G}| < s \leqslant a$. As $a' \in \mathcal{I}^+(\mathcal{A}_\delta)$ one can find $a \in [a']^\omega$ such that $\{a\} \cup \mathcal{A}_\delta \restriction (\mathcal{F} \cup \mathcal{G})$ is almost disjoint family. Finally, one can easily check that such a works, i.e. $\mathcal{A}_\delta \cup \{a\}$ is almost disjoint family.

Completely separable MAD family from $\mathfrak{s} \leqslant \mathfrak{a}$

Final inductive construction

Enumerate $\{b_\delta : \delta < \mathfrak{c}\} = [\omega]^\omega$. Use *Main Lemma* through all $\mathfrak{c}$ - at stage $\delta < \mathfrak{c}$ for $b = b_\delta$ if $b_\delta \in \mathcal{I}^+(\mathcal{A}_\delta)$ and $b = \omega$ otherwise.

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This gives families $\{\sigma_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^{<s}$, $\{a_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ such that for all $\alpha < \mathfrak{c}$ it holds $a_\alpha \in \mathcal{I}_{\sigma_\alpha}$ and $a_\alpha \subseteq b_\alpha$ if $b_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$.

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Why it works? Given any $b \in \mathcal{I}^+(\mathcal{A}_\mathfrak{c}) = \bigcap\{\mathcal{I}^+(\mathcal{A}_\alpha) : \alpha < \mathfrak{c}\}$ choose $\delta < \mathfrak{c}$ with $b = b_\delta$. Then by tha above construction $b_\delta \supseteq a_\delta \in \mathcal{A}_\mathfrak{c}$, so completely separable MAD family is cooked.

Open Problem

How to get rid of $P(a, s)$ from the third case of Shelah's proof?

Conclusion

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Quest for splitting families

Use $b \leqslant a < s$ to find a special splitting family related (somehow) to b and prove analogons of Splitting Lemma and Main Lemma. While H.Mildenberger, D.Raghavan, J.Steprāns used $s_{\omega,\omega}$ to remove the assumption $U(s)$ from the second case, the framework of $s(\mathcal{A})$'s does not seemed to be sufficient for the removing $P(a, s)$.

THANKS !!!

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