# Cardinal Invariants and the P-Ideal Dichotomy

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# A Gentle Introduction

## Definition (Eventually Dominating)

- For $f, g \in \omega^\omega$ we say $f <^* g$ if there is some $n \in \omega$ such that for every $k \geq n$, $f(k) < g(k)$.
- Let $\chi(f, g)$ be the minimum $n$ for which the above statement holds.

Do we need a reminder of $\mathfrak{b}$ and $\mathfrak{d}$?

## Definition (P-ideal)

A *P-ideal* is an ideal $\mathcal{I}$ of subsets of some set $X$ such that if $\{A_i : i \in \omega\} \subseteq \mathcal{I}$, then there is some $A \in \mathcal{I}$ such that for each $i \in \omega$, $A_i \subseteq^* A$. (i.e. $A_i \setminus A$ is finite).

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## Example (A not so interesting P-ideal when $\mathfrak{b} < \mathfrak{d}$)

- Suppose $\mathfrak{b} < \mathfrak{d}$
- Let $\langle f_\alpha : \alpha \in \mathfrak{b} \rangle$ be an increasing unbounded sequence in $(\omega^\omega, \leq^*)$.
- It is **not** cofinal, so there is some $f \in \omega^\omega$ such that for all $\alpha \in \mathfrak{b}$, $f \not\leq^* f_\alpha$.
- Let $b_\alpha := \{n \in \omega : f(n) < f_\alpha(n)\}$
- Let $I$ be the ideal generated by $\{b_\alpha : \alpha \in \mathfrak{b}\}$.
- Then for $\alpha < \beta$ it follows from $f_\alpha \leq^* f_\beta$ that $b_\alpha \subseteq^* b_\beta$.
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## Definition (The P-ideal Dichotomy (PID))

The *P-ideal dichotomy* is the following statement:
For every P-ideal $\mathcal{I}$ of countable subsets of some uncountable set $S$, either

1. there is an uncountable $A \subseteq S$ such that $[A]^\omega \subseteq \mathcal{I}$, or
2. $S$ can be decomposed into countably many sets orthogonal to $\mathcal{I}$.

# Gaps and P-ideals

## Definition (Gaps)

- A sequence $\langle\{f_\alpha : \alpha \in \kappa\}, \{g_\beta : \beta \in \lambda\}\rangle$ is a *pregap* if $f_{\alpha_1} <^* f_{\alpha_2} <^* g_{\beta_2} <^* g_{\beta_1}$ for all $\alpha_1 < \alpha_2 < \kappa$ and all $\beta_1 < \beta_2 < \lambda$.
- A pregap as defined above is a (unfilled) gap if there is no such $h \in \omega^\omega$ such that $f_\alpha <^* h <^* g_\beta$ for all $\alpha \in \kappa$ and all $\beta \in \lambda$.

## Example (A P-ideal from a gap)

Let $\langle\{f_\alpha : \alpha \in \kappa\}, \{g_\beta : \beta \in \lambda\}\rangle$ be a (unfilled) gap, where $\kappa$ and $\lambda$ are regular and uncountable. Then define the ideal $\mathcal{I} \subseteq [\kappa]^\omega$ by the following. $A \in \mathcal{I}$ if and only if there exists an $\beta \in \lambda$ such that for all $n \in \omega$ the set

$$\{\alpha \in A : \chi(f_\alpha, g_\beta) < n\}$$

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## Reminder

$A \in \mathcal{I} \subseteq [\kappa]^\omega \iff \exists\beta\ \forall n\ \{\alpha \in A : \chi(f_\alpha, g_\beta) < n\}$ *is finite*

## Proof that $\mathcal{I}$ is a P-ideal.

- Suppose $\{A_i : i \in \omega\} \subseteq \mathcal{I}$ has witnesses $\{g_{\beta_i} : i \in \omega\}$.
- Let $\beta := \sup\{\beta_i : i \in \omega\}$.
- Notice $\{\alpha \in A_i : \chi(f_\alpha, g_\beta) < n\}$ $\subseteq \{\alpha \in A_i : \chi(f_\alpha, g_{\beta_i}) < \max\{n, \chi(g_\beta, g_{\beta_i})\}\}$.
- Define $A_i' := A_i \setminus \{\alpha \in A_i : \chi(f_\alpha, g_\beta) < i\}$.
- Define $A := \bigcup_{i\in\omega} A_i'$.
- We claim $A_i \subseteq^* A$ for all $i$, and $A \in \mathcal{I}$.

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## Lemma

*Applying the PID to this ideal gives us that* $\kappa = \lambda = \aleph_1$.

## Proof under the first alternative of PID.

- $S := \{\alpha_\gamma : \gamma \in \omega_1\}$ such that $[S]^\omega \in \mathcal{I}$.
- For each $\delta \in \omega_1$, let $g_{\beta_\delta}$ witness $A_\delta := \{\alpha_\gamma : \gamma \in \delta\} \in \mathcal{I}$.
- Claim: $\langle \{f_{\alpha_\gamma} : \gamma \in \omega_1\}, \{g_{\beta_\delta} : \delta \in \omega_1\} \rangle$ is unfilled.
- $f_{\sup\{\alpha_\gamma : \gamma \in \omega_1\}+1}$ or $g_{\sup\{\beta_\delta : \delta \in \omega_1\}+1}$ would fill this gap if they existed.
- $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda) = \aleph_1$.

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- $S \subseteq \kappa$ cofinal in $\kappa$ and orthogonal to $\mathcal{I}$.
- $f(n) := \sup\{f_\alpha(n) : \alpha \in S\}$. (Possibly bounded by $g_0$).
- There is a $\beta$ such that $f \not<^* g_\beta$. Thus for each $i \in \omega$ there is an $n_i > i$ with $g_\beta(n_i) \leq f(n_i)$.
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- Claim: $A := \{\alpha_i : i \in \omega\} \in [S]^\omega \cap \mathcal{I}$.
- Contradiction $\Rightarrow\Leftarrow$

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$A \in \mathcal{I} \subseteq [\kappa]^\omega \iff \exists\beta\ \forall n\ \{\alpha \in A : \chi(f_\alpha, g_\beta) < n\}$ *is finite*

## Lemma

*Applying the PID to this ideal gives us that* $\kappa = \lambda = \aleph_1$.

## Proof under the second alternative of PID.

- $S \subseteq \kappa$ cofinal in $\kappa$ and orthogonal to $\mathcal{I}$.
- $f(n) := \sup\{f_\alpha(n) : \alpha \in S\}$. (Possibly bounded by $g_0$).
- There is a $\beta$ such that $f \not<^* g_\beta$, Thus for each $i \in \omega$ there is an $n_i > i$ with $g_\beta(n_i) \leq f(n_i)$.
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- Contradiction $\Rightarrow\Leftarrow$

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*PID* $\Rightarrow \mathfrak{p} = \mathfrak{t}$.

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It has long been known that $\mathfrak{p} \leq \mathfrak{t}$.
In the paper "A Comment on $\mathfrak{p} < \mathfrak{t}$" (2009), Shelah showed that if $\mathfrak{p} < \mathfrak{t}$, then there is an uncountable $\kappa < \mathfrak{p}$ and a $(\kappa, \mathfrak{p})$-gap. □

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# PID and $\mathfrak{b}$

## Remark

*We know already:*

- $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b}$
- $\mathfrak{p} = \aleph_1 \Rightarrow \mathfrak{p} = \mathfrak{t}$.

*So even without gaps, we could have known that PID $\Rightarrow \mathfrak{p} = \mathfrak{t}$ if only we knew that PID $\Rightarrow \mathfrak{b} \leq \aleph_2$.*

## Definition

- For $g \in \omega^\omega$, let $(< g)$ denote the set $\{f \in \omega^\omega : f <^* g\}$
- We say $A \subseteq (< g)$ is *near* $g$ if for every $n$, the set $\{f \in A : \chi(f, g) < n\}$ is finite.

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## Theorem

*$\mathcal{I}_g$ is a P-ideal.*

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For $\{A_i : i \in \omega\} \subseteq \mathcal{I}_g$, the set

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## Definition (A P-ideal $\mathcal{I}$ on $\omega_2$ if $\mathfrak{b} > \omega_2$)

Let us assume now that $\mathfrak{b} > \omega_2$, in which case, we can find a $<^*$-increasing sequence of functions $\langle f_\xi : \xi \in \omega_2 \rangle$.
Now we say $X \in \mathcal{I}$ if $X$ is countable, and for some $\nu \in \omega_2$, for all $\mu \geq \nu$ we have $\{f_\xi : \xi \in X\} \in \mathcal{I}_{f_\mu}$.

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**Theorem (Todorcevic)**

$PID \Rightarrow \mathfrak{b} \leq \omega_2$.

**Proof (part 1).**

Using the ideal $\mathcal{I}$ previously described, assume there is some uncontable $A \subseteq \omega_2$ with $[A]^\omega \subseteq \mathcal{I}$. (We may assume that $A$ has order type $\omega_1$.) Then we can find a single $\nu < \omega_2$ such that every initial segment of $A' \subseteq A$ is in $\mathcal{I}_{f_\nu}$. However, we can find an uncountable set $B \subseteq A$ such that for all $\alpha \in B$, $\chi(f_\alpha, f_\nu)$ is constant. But now take any intial segment $A' \subseteq A$ such that $|B \cap A'| = \omega$. Clearly then $A'$ is not near $f_\nu$, creating a contradiction. □

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**Proof (part 2).**

Otherwise, $\omega_2$ is the countable union of sets orthogonal to $\mathcal{I}$. So in particular there is a cofinal $E \subseteq \omega_2$ orthogonal to $\mathcal{I}$.

Choose some $g \in \omega^\omega$ such that $f_\xi \leq^* g$ for all $\xi \in E$. If the set $B := \{n \in \omega : \sup_{\xi \in E} f_\xi(n) = \omega\}$ was infinite, choose for each $n \in B$ a $\xi_n \in \omega_2$ such that $g(n) < f_{\xi_n}(n)$. Like before, $\{f_{\xi_n} : n \in \omega\} \in \mathcal{I}$.

So $B$ is finite, and thus for $n$ large enough $s(n) := \sup_{\xi \in E} f_\xi(n)$ is finite. Define now $s'(n) := s(n) - 1$. Notice that we have for each $\xi \in E$ that $f_\xi <^* s'$. We can still however find for each $n$ (large enough), a $\xi_n \in E$ such that $s'(n) < f_{\xi_n}(n)$. Again, as before, $\{f_{\xi_n} : n \in \omega\} \in \mathcal{I}$. □

# Forcing the PID

## Definition (Forcing the PID (Todorcevic 1999))

Let $\mathcal{I}$ be a P-ideal on some ordinal $\theta$ such that $\theta$ cannot be decomposed into countably many sets orthogonal to $\mathcal{I}$, but every smaller ordinal can. Then we define the forcing poset $\mathbb{P} = \mathbb{P}_{\mathcal{I}}$ by the following: $p \in \mathbb{P}$ when $p = \langle x_p, \mathfrak{X}_p \rangle$ and

- $x_p$ is a countable subset of $\theta$, and
- $\mathfrak{X}_p$ is a countable collection of cofinal subsets of $\langle [\mathcal{I}]^\omega, \subseteq \rangle$.

Now for each $a \in [\mathcal{I}]^\omega$, choose a fixed $m_a \in \mathcal{I}$ such that $b \subseteq^* m_a$ for every $b \in A$. Then we say $q \leq p$ ($q$ extends $p$) when

- $x_q$ end-extends $x_p$
- $\mathfrak{X}_p \subseteq \mathfrak{X}_q$, and
- for every $X \in \mathfrak{X}_p$, $\{a \in X : x_q \setminus x_p \subseteq m_a\} \in \mathfrak{X}_q$ (and is cofinal in $[\mathcal{I}]^\omega$).

## Remark

*This forcing is proper and forces that there is an uncountable subset $A$ of $\theta$ such that $[A]^\omega \subseteq \mathcal{I}$. It also adds no new reals, even when iterated with countable support.*

## Implications

- *PFA $\Rightarrow$ PID*
- *If there is a supercompact cardinal $\kappa$, we can iterate over $\kappa$ to force both the PID and the GCH.*
- *$\text{PID}_{\omega_1}$ can be forced by only iterating $\omega_2$ many posets.*

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## Goals

- *Using a supercompact cardinal, force PID and $\mathfrak{b} < \mathfrak{d}$.*
- *More generally, force PID with all configurations of Cichon's diagram. (where $\mathfrak{c} = \aleph_1$)*

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*Given the PID,*

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