On existence of independent sets in partially ordered sets

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Winter School Hejnice 2013
The strong sequences method was introduced by B. A. Efimov, as a useful method for proving famous theorems in dyadic spaces like: Marczewski theorem on cellularity, Shanin theorem on a calibre, Esenin-Volpin theorem, Erdös-Rado theorem and others.
Let \( T \) be an infinite set. Denote *the Cantor cube* by

\[
D^T = \{ p: p: T \rightarrow \{0, 1\} \}.
\]

For \( s \subset T, i: s \rightarrow \{0, 1\} \) it will be used the following notation

\[
H^i_s = \{ p \in D^T : p|s = i \}.
\]

Efimov defined strong sequences in the subbase \( \{ H^i_\alpha : \alpha \in T \} \) of the Cantor cube and proved the following
Theorem (Efimov)

Let $\kappa$ be a regular, uncountable cardinal number. In the space $D^T$ there is not a strong sequence

$$
(\{H^i_\alpha : \alpha \in v_\xi\}, \{H^i_\beta : \beta \in w_\xi\}) ; \; \xi < \kappa
$$

such that $|w_\xi| < \kappa$ and $|v_\xi| < \omega$ for each $\xi < \kappa$. 
Let $X$ be a set, and $B \subseteq P(X)$ be a family of non-empty subsets of $X$ closed with respect to finite intersections. Let $S$ be a finite subfamily contained $B$. A pair $(S, H)$, where $H \subseteq B$, will be called \textit{connected} if $S \cup H$ is centered.
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**Definition (Turzański)**

A sequence $(S_\phi, H_\phi); \phi < \alpha$ consisting of connected pairs is called a strong sequence if $S_\lambda \cup H_\phi$ is not centered whenever $\lambda > \phi$. 
Theorem (Turzański)

If for $B \subset P(X)$ there exists a strong sequence $S = (S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ such that $|H_\phi| \leq \kappa$ for each $\phi < (\kappa^\lambda)^+$ then there exists a strong sequence $(S_\phi, T_\phi); \phi < \lambda^+$, where $|T_\phi| < \omega$ for each $\phi < \lambda^+$
In 2008
J. Jureczko, M. Turzański,
From a Ramsey-Type Theorem To Independence,
Definition

We say that a family of sets \( \mathcal{S} \) fulfills condition (I) if for all \( S_0, S_1, S_2 \in \mathcal{S} \), if \( S_0 \cap S_1 = \emptyset \) and \( S_0 \cap S_2 = \emptyset \) then either \( S_1 \cap S_2 = \emptyset \) or \( S_1 \subset S_2 \) or \( S_2 \subset S_1 \).
Definition

We say that a family of sets $\mathcal{I}$ fulfills condition (I) if for all $S_0, S_1, S_2 \in \mathcal{I}$, if $S_0 \cap S_1 = \emptyset$ and $S_0 \cap S_2 = \emptyset$ then either $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$ or $S_2 \subset S_1$.

Definition

We say that a family of sets $\mathcal{I}$ fulfills condition $(T(\kappa))$ if for each set $U \in \mathcal{I}$ there is

$$|\{V \in \mathcal{I} : V \subset U\}| < \kappa$$
Definition

A family \( \{ (A^0_\xi, A^1_\xi) : \xi < \alpha \} \) of ordered pairs of subsets of \( X \) such that \( A^0_\xi \cap A^1_\xi = \emptyset \) for \( \xi < \alpha \) is called a weakly independent family (of length \( \alpha \)) if for each \( \xi, \zeta < \alpha \) with \( \xi \neq \zeta \) we have \( A^i_\xi \cap A^j_\zeta \neq \emptyset \), where \( i, j \in \{0, 1\} \).
Definition

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Theorem

Let \( \mathcal{S} \) be a family of sets which has the following properties:
(i) \( \mathcal{S} \) fulfills condition (I);
(ii) \( \mathcal{S} \) fulfills condition \( (T(\kappa)) \);
(iii) for each \( U \in \mathcal{S} \) there is \( X \setminus U \in \mathcal{S} \).

Then for each regular cardinal number \( \kappa \) such that \( |\mathcal{S}| \geq \kappa > c(\mathcal{S}) \) there exists a weakly independent family in \( \mathcal{S} \) of cardinality \( \kappa \).
Definition

A family of sets $\mathcal{S}$ is said to be binary if for each finite subfamily $\mathcal{M} \subset \mathcal{S}$ with $\bigcap \mathcal{M} = \emptyset$ there exist $A, B \in \mathcal{M}$ such that $A \cap B = \emptyset$. 
**Definition**

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**Definition**

A family \( \{(A_\xi, B_\xi): \xi < \alpha\} \) of ordered pairs of subsets of $X$, such that $A_\xi \cap B_\xi = \emptyset$ for $\xi < \alpha$ is called an independent family (of length $\alpha$) if for each finite subset $F \subset \alpha$ and each function $i: F \to \{-1, +1\}$ we have

$$\bigcap \{i(\xi)A_\xi: \xi \in F\} \neq \emptyset$$

(where $(+1)A_\xi = A_\xi, (-1)A_\xi = B_\xi$).
Corollary

Let $X$ be a compact zero-dimensional space. Let $\mathcal{S}$ be a family consisting of clopen sets which has the following properties:

(i) $\mathcal{S}$ is a binary family;
(ii) $\mathcal{S}$ fulfills condition (I);
(iii) $\mathcal{S}$ fulfills condition $(T(\kappa))$;
(iv) for each $U \in \mathcal{S}$ the set $X \setminus U \in \mathcal{S}$.

Then for each regular cardinal number $\kappa$ such that $|\mathcal{S}| \geq \kappa > c(\mathcal{S})$ there exists an independent family in $\mathcal{S}$ of cardinality $\kappa$. 
Let \((X, r)\) be a set with relation \(r\).

We say that \(a\) and \(b\) are comparable if \((a, b) \in r\) or \((b, a) \in r\).

We say that \(a\) and \(b\) are compatible if there exists \(c\) such that \((a, c) \in r\) and \((b, c) \in r\).

(We say then that \(a\) and \(b\) have an upper bound).

If each of two elements in a set \(A \subset X\) are compatible, then \(A\) is an upper directed set.

A set \(A\) is \(\kappa\)-upper directed if every subset of \(X\) of cardinality less than \(\kappa\) has an upper bound, i.e. for each \(B \subset X\) with \(|B| < \kappa\) there exists \(a \in A\) such that \((b, a) \in r\) for all \(b \in B\).
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Definition

Let \((X, r)\) be a set with relation \(r\).
A sequence \((S_\phi, H_\phi); \phi < \alpha\) where \(S_\phi, H_\phi \subset X\) and \(S_\phi\) is finite is called a strong sequence if
1° \(S_\phi \cup H_\phi\) is \(\omega\)-upper directed
2° \(S_\beta \cup H_\phi\) is not \(\omega\)-upper directed for \(\beta > \phi\).
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We say that $(X, r)$ has $Q(\kappa)$-property iff for all $x, y \in X$ if $x \parallel y$ then
$$|\{z \in X : x \perp z \lor z \perp y\}| = \kappa.$$
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• We say that $\mathcal{L} \subset X$ is a *chain* if any $a, b \in \mathcal{L}$ are comparable.

• We say that a set $\mathcal{A} \subset X$ is called an *antichain* if any two distinct elements $a, b \in \mathcal{A}$ are incompatible.

• The minimal cardinal $\kappa$ such that every antichain in $X$ has size less than $\kappa$ is *saturation of $X$* and denote it by $\text{sat}(X)$.
Definition

A sequence of ordered pairs \( \{(x^0_\alpha, x^1_\alpha)\} \) where \( x^0_\alpha \perp x^1_\alpha \) is said to be an independent set if for each finite set \( F \subseteq \kappa \) and for each function \( i : F \to \{0, 1\} \) the set \( \{x^{i(\alpha)}_\alpha : \alpha \in F\} \) is \( \omega \)– upper directed.

Theorem

Let \( \kappa \) be a regular cardinal number. Let \( (X, r) \) be a set with relation which has \( A(\omega) \)– and \( Q(\omega) \)–property. If \( |X| = \kappa > sat(X) \) then there exists an independent set in \( X \) of cardinality \( \kappa \).
Definition

A sequence of ordered pairs \( \{(x_0^\alpha, x_1^\alpha)\} \) where \( x_0^\alpha \perp x_1^\alpha \) is said to be an independent set if for each finite set \( F \subset \kappa \) and for each function \( i: F \to \{0, 1\} \) the set \( \{x_\alpha^{i(\alpha)}: \alpha \in F\} \) is \( \omega \)-upper directed.

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Definition

A cardinal $\kappa$ is a *calibre* for $X$ if $\kappa$ is infinite and every set $A \in [X]^\kappa$ has a chain of size $\kappa$. 

Note: Each calibre is a precalibre but the inverse theorem is not true.
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**Theorem**

Let \((X, r)\) be a set with relation \(r\). Then each regular cardinal number \(\tau > s\) is a precalibre for \(X\).
Theorem

Let $\tau$ be a cardinal number. Let $(X, r)$ be a set with relation and $\tau^+$ be a precalibre of $X$. If $|X| > 2^\tau$, then there exists an independent set of cardinality $\tau^+$. 
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Let \((X, r)\) be a set with relation. Let \(\tau\) be a regular cardinal number which is a precalibre for \(X\). Then \(i > \tau > s\).
Theorem

Let $\kappa \geq \omega$ and $(X, r)$ be a set with relation of cardinality at least $\kappa$. If $(X, r)$ has $A(\kappa)$- and $Q(\kappa)$-property then there exists a set $A \subset X$ of cardinality $\kappa$ which is both a maximal $\kappa$-independent set and a maximal independent set.
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Let $\kappa \geq \omega$ and $(X, r)$ be a set with relation of cardinality at least $\kappa$. If $(X, r)$ has $A(\kappa)$- and $Q(\kappa)$-property then there exists a set $A \subset X$ of cardinality $\kappa$ which is both a maximal $\kappa$-independent set and a maximal independent set.

Corollary
Let $\kappa \geq \omega$ and $(X, r)$ be a set with relation of cardinality at least $\kappa$. If $(X, r)$ has $A(\kappa)$- and $Q(\kappa)$-property, then $i_\kappa = i$. 
Theorem

Let $(X, r)$ be a set with relation $r$. Then $s \geq \text{sat}(X)$. 

Corollary

Let $(X, r)$ be a set with relation. Let $\tau$ be a precalibre of $X$.

Then $i \kappa > \tau > s \geq \text{sat}(X)$. 

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References


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