On skeletally factorizable spaces

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Skeletal maps

A map $s\colon X \to Y$ between topological spaces is called skeletal if for each nowhere dense subset $A \subset Y$ the preimage $s^{-1}(A)$ is nowhere dense in X.

This is equivalent to saying that:

for each non-empty open set $U \subset X$ the closure $\overline{f(U)}$ has non-empty interior in Y.

Examples

- each open map is skeletal;
- the identity map from the Sorgenfrey line onto the real line is skeletal but not open.

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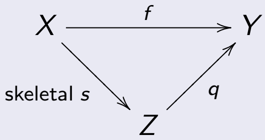
Skeletally factorizable spaces

Definition

A topological space X is defined to be skeletally factorizable if each map $f: X \to Y$ to a second countable space Y can be written as the composition $f = q \circ s$ of a skeletal map $s: X \to Z$ onto a second countable space Z and a map $q: Z \to Y$:



Example

- A Tychonoff space X is skeletally factorizable if the set D_X of isolated points of X is countable and dense in X.
- Any compactification $c\mathbb{N}$ of $\mathbb{N}$ is skeletally factorizable.

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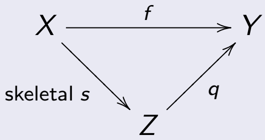

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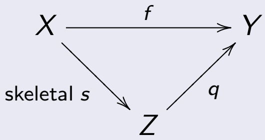

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Openly factorizable spaces

Definition (Banakh-Dimitrova)

A topological space X is defined to be openly factorizable if each map $f\colon X \to Y$ to a second countable space Y can be written as the composition $f = q \circ p$ of an open map $p\colon X \to Z$ onto a second countable space Z and a map $q\colon Z \to Y$:

Each openly factorizable space is skeletally factorizable.

Example

- The ordinal segment $[0, \omega_1]$ is openly factorizable.
- The connected long line $\overline{[0, \omega_1]}$ is not skeletally factorizable.
- The Stone-Čech compactification $\beta\mathbb{N}$ of integers is skeletally factorizable but not openly factorizable.

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Theorem (Banakh-Dimitrova, 2010)

The Stone-Čech compactification βX of a Tychonoff space X is openly factorizable if and only if X is pseudocompact and openly factorizable.

Theorem (M., 2012)

For a Tychonoff space X the following conditions are equivalent:

1) *X is skeletally factorizable;*
2) *there is a skeletally factorizable space Y with $X \subset Y \subset \beta X$;*
3) *each space Y, $X \subset Y \subset \beta X$, is skeletally factorizable;*
4) *βX is skeletally factorizable.*

Corollary

The Stone-Čech compactification βD of a discrete space D is skeletally factorizable if and only if $|D| \leq \aleph_0$.

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Scattered skeletally and openly factorizable compacta

A space X is scattered if each subspace of X has an isolated point.

A point x of topological space X is a P-point if for any neighborhoods $U_n \subset X$, $n \in \omega$, the intersection $\bigcap_{n\in\omega} U_n$ is neighborhood of x in X.

Theorem (Banakh-Dimitrova, 2010)

A scattered (linearly ordered) compact space X is openly factorizable if (and only if) each point $x \in X$ is either a G_δ-point or a P-point.

Theorem (Banakh-M., 2012)

A scattered compact Hausdorff space X is skeletally factorizable iff each non-P-point $x \in X$ lies in G_δ-subset $G \subset X'$ of X, where X' is the set of all non-isolated points of X.

Corollary

- *The segment of ordinals $[0, \kappa]$ is openly factorizable.*
- *The one-point compactification αD of an uncountable discrete space D is not skeletally factorizable.*

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Spectral characterization of skeletally factorizable spaces

Theorem (M., 2012)

A Lindelöf topological space X is skeletally factorizable iff X is the limit space $\lim \mathcal{S}$ of an inverse spectrum $\mathcal{S} = \{X_\alpha, \pi_\alpha^\gamma, A\}$ over an ω-directed index set A such that for every $\alpha \in A$ the space X_α is second countable and the limit projection $\pi_\alpha \colon X \to X_\alpha$ is skeletal and surjective.

Preservation of skeletally factorizable spaces by functors

A functor F : **Comp** $\to$ **Comp** is skeletal if for each surjective skeletal map $f\colon X \to Y$ between compact Hausdorff spaces the map $Ff\colon FX \to FY$ is skeletal.

Theorem (M., 2012)

Each skeletal epimorphic continuous functor F : **Comp** $\to$ **Comp** *preserves the class of skeletally factorizable compacta.*

Theorem (Banakh-Kucharski-M., 2012)

A normal functor F : **Comp** $\to$ **Comp** *is skeletal if and only if for every metrizable zero-dimensional compact space Z the map $Z \oplus 2 \to Z \oplus 1$ is skeletal.*

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Skeletally and openly generated spaces.

Definition

A compact space X is called skeletally generated (resp. openly generated) if X is homeomorphic to the limit space of a continuous spectrum $\{X_\alpha, p_\alpha^\beta, A\}$ indexed by a ω-complete directed set A and consisting of second countable spaces X_α and skeletal (open) bonding projections p_α^β.

Theorem (Daniel-Kune-Zhou, 1994)

Each skeletally generated compact space has countable cellularity.

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Relations between various classes of compacta

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\Downarrow & & \Downarrow \\
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\end{array}
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None of these implications can be reversed:

- $\beta\mathbb{N}$ is skeletally generated but not openly factorizable.
- $[0, \omega_1]$ is openly factorizable but not skeletally generated.

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