On isomorphisms and embeddings of $C(K)$ spaces

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Hejnice, January/February 2013
Preliminaries

\begin{equation}
\text{and } L \text{ always stand for compact Hausdorff spaces.}
\end{equation}

For a given \( K \), \( C(K) \) is the Banach space of all continuous real-valued functions \( f : K \to \mathbb{R} \), with the usual norm:

\[ ||g|| = \sup_{x \in K} |f(x)|. \]

A linear operator \( T : C(K) \to C(L) \) is an isomorphic embedding if there are \( M, m > 0 \) such that for every \( g \in C(K) \)

\[ m \cdot ||g|| \leq ||Tg|| \leq M \cdot ||g||. \]

If \( M \) is the least constant with such a property then \( M = ||T|| \), likewise \( m = \frac{1}{||T^{-1}||} \).

Isomorphic embedding \( T : C(K) \to C(L) \) which is onto is called an isomorphism; we then write \( C(K) \sim C(L) \).
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A linear operator $T : C(K) \to C(L)$ is an isomorphic embedding if there are $M, m > 0$ such that for every $g \in C(K)$, $m \cdot \|g\| \leq \|Tg\| \leq M \cdot \|g\|$. If $M$ is the least constant with such a property then $M = \|T\|$, likewise $m = \frac{1}{\|T^{-1}\|}$. Isomorphic embedding $T : C(K) \to C(L)$ which is onto is called an isomorphism; we then write $C(K) \cong C(L)$. 

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Theorem

Under CH, for every $K$ of weight $\leq \aleph_1$, $C(K)$ embeds isometrically into $C(\omega^\ast)$ (which itself is isometric to $l_\infty/\ell_0$).

Dow & Hart: Consistently, the measure algebra does not embed into $P(\omega)/\text{fin}$, so its Stone space $S$ is not an image of $\omega^\ast$.

On the other hand, $C(S) \equiv L_\infty[0,1] \sim l_\infty \equiv C(\beta\omega)$ embeds into $C(\omega^\ast)$.

Todorˇ cević (2011) proved that, consistently, there is a "small" compact $K$ such that $C(K)$ does not embed into $C(\omega^\ast)$, cf. Krupski-Marciszewski (2012).
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*Under CH, for every $K$ of weight $\leq c$, $C(K)$ embeds isometrically into $C(\omega^*)$ (which itself is isometric to $l_\infty/c_0$).*
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**C(K) spaces for nonmetrizable K**

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Some ancient results

- **Banach-Stone:**
  \[ C(K) \text{ is isometric to } C(L) \text{ then } K \cong L. \]

- **Amir, Cambern:**
  \[ T : C(K) \to C(L) \text{ is an isomorphism with } ||T|| \cdot ||T^{-1}|| < 2 \text{ then } K \cong L. \]

- **Jarosz (1984):**
  \[ T : C(K) \to C(L) \text{ is an embedding with } ||T|| \cdot ||T^{-1}|| < 2 \text{ then } K \text{ is a continuous image of some compact subspace of } L. \]

- **Miljutin:**
  \[ K \text{ is an uncountable metric space then } C(K) \sim C([0,1]). \]
  In particular
  \[ C(2^{\omega}) \sim C([0,1]); \quad C([0,1] \cup \{2\}) \sim C([0,1]). \]
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Some ancient problems

Problem
For which spaces $K$, $C(K) \sim C(K+1)$?

Here $K+1$ denotes $K$ with one additional isolated point.

This is so if $K$ contains a nontrivial converging sequence:

$$C(K) = c_0 \oplus X \sim c_0 \oplus X \oplus \mathbb{R} \sim C(K+1).$$

Note that $C(\beta\omega) \sim C(\beta\omega+1)$ (because $C(\beta\omega) = l_\infty$) though $\beta\omega$ has no converging sequences.

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For which spaces $K$ there is a totally disconnected $L$ such that $C(K) \sim C(L)$?
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Two peculiar compacta

Koszmider (2004):
There is a compact connected space $K$ such that every bounded operator $T : C(K) \to C(K)$ is of the form $T = g \cdot I + S$, where $S : C(K) \to C(K)$ is weakly compact.

Consequently, $C(K) \not\sim C(K + 1)$, and $C(K)$ is not isomorphic to $C(L)$ with $L$ totally disconnected.

Aviles-Koszmider (2011):
There is a space $K$ which is not Radon-Nikodym compact but is a continuous image of an RN compactum; it follows that $C(K)$ is not isomorphic to $C(L)$ with $L$ totally disconnected.

Problem (Argyros & Arvanitakis)
Let $K$ be a compact convex subset of some Banach space which is not metrizable. Can $C(K)$ be isomorphic to $C(L)$, where $L$ is totally disconnected?
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Problem (Argyros & Arvanitakis)

*Let $K$ be a compact convex subset of some Banach space which is not metrizable. Can $C(K)$ be isomorphic to $C(L)$, where $L$ is totally disconnected?*
Some questions

Suppose that $C(K)$ and $C(L)$ are isomorphic. How is $K$ topologically related to $L$?

Suppose that $C(K)$ can be embedded into $C(L)$, where $L$ has some property $P$. Does $K$ have property $P$?
Suppose that \( C(K) \) and \( C(L) \) are isomorphic. How \( K \) is topologically related to \( L \)?
Some questions

- Suppose that $C(K)$ and $C(L)$ are isomorphic. How $K$ is topologically related to $L$?
- Suppose that $C(K)$ can be embedded into $C(L)$, where $L$ has some property $\mathcal{P}$. Does $K$ have property $\mathcal{P}$?
Results on positive embeddings

Let $T : C(K) \rightarrow C(L)$ be a positive isomorphic embedding. Then there is $p \in \mathbb{N}$ and a finite valued mapping $\phi : L \rightarrow \mathbb{K}^p$ which is onto ($\bigcup y \in L \phi(y) = K$) and upper semicontinuous (i.e. \{y : \phi(y) \subseteq U\} \subseteq L is open for every open $U$).

Corollary
If $C(K)$ can be embedded into $C(L)$ by a positive operator then $\tau(K) \leq \tau(L)$ and if $L$ is Fréchet (or sequentially compact) then $K$ is Fréchet (sequentially compact).

Remark:
$p$ is the integer part of $\|T\| \cdot \|T^{-1}\|$. 

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Results on positive embeddings

An embedding $T : C(K) \to C(L)$ is **positive** if $C(K) \ni g \geq 0$ implies $Tg \geq 0$. 

Theorem

Let $T : C(K) \to C(L)$ be a positive isomorphic embedding. Then there is $p \in \mathbb{N}$ and a finite valued mapping $\phi : L \to [K]/x_{\leq p}$ which is onto ($\bigcup y \in L \phi(y) = K$) and upper semicontinuous (i.e. $\{ y : \phi(y) \subseteq U \} \subseteq L$ is open for every open $U \subseteq K$).

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If $C(K)$ can be embedded into $C(L)$ by a positive operator then $\tau(K)/x_{\leq p} \geq \tau(L)$ and if $L$ is Frechet (or sequentially compact) then $K$ is Frechet (sequentially compact).

Remark:

$p$ is the integer part of $||T|| \cdot ||T^{-1}||$. 

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Isomorphisms of $C(K)$ spaces
Results on positive embeddings

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**Theorem**

Let $T : C(K) \to C(L)$ be a positive isomorphic embedding. Then there is $p \in \mathbb{N}$ and a finite valued mapping $\varphi : L \to [K]^{\leq p}$ which is onto ($\bigcup_{y \in L} \varphi(y) = K$) and upper semicontinuous (i.e. $\{y : \varphi(y) \subseteq U\} \subseteq L$ is open for every open $U \subseteq K$).

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If $C(K)$ can be embedded into $C(L)$ by a positive operator then $\tau(K) \leq \tau(L)$ and if $L$ is Frechet (or sequentially compact) then $K$ is Frechet (sequentially compact).

Remark: $p$ is the integer part of $\|T\| \cdot \|T^{-1}\|$. 
Theorem
Suppose that there is an operator $T : C(K) \to C(L)$ such that $T$ is either positive isomorphic embedding or an arbitrary isomorphism. Then there is nonempty open $U \subseteq K$ such that $U$ is a continuous image of some compact subspace of $L$. In fact the family of such $U$ forms a $\pi$-base in $K$.

Corollary
If $C([0,1]) \sim C(L)$ then $L$ maps continuously onto $[0,1]$. 
Main result

Theorem

Suppose that there is an operator $T : C(K) \to C(L)$ such that $T$ is either positive isomorphic embedding or an arbitrary isomorphism.

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Isomorphisms of $C(K)$ spaces 
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Corollary
If $C[0, 1]^\kappa \sim C(L)$ then $L$ maps continuously onto $[0, 1]^\kappa$. 
This is equivalent to saying that $C(K)$ contains a point-countable family separating points of $K$.

Problem Suppose that $C(K) \sim C(L)$, where $L$ is Corson compact. Must $K$ be Corson compact?

The answer is 'yes' under $\text{MA}(\omega_1)$.

Theorem If $C(K) \sim C(L)$, where $L$ is Corson compact then $K$ has a $\pi$-base of sets having Corson compact closures. In particular, $K$ is itself Corson compact whenever $K$ is homogeneous.
$K$ is **Corson compact** if $K \hookrightarrow \Sigma(\mathbb{R}^\kappa)$ for some $\kappa$, where

$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$
Corson compacta

\( K \) is **Corson compact** if \( K \hookrightarrow \Sigma(\mathbb{R}^\kappa) \) for some \( \kappa \), where

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*Suppose that $C(K) \sim C(L)$, where $L$ is Corson compact. Must $K$ be Corson compact?*
Corson compacta

If $K$ is Corson compact if $K \hookrightarrow \Sigma(\mathbb{R}^\kappa)$ for some $\kappa$, where
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\Sigma(\mathbb{R}^\kappa) = \{ x \in \mathbb{R}^\kappa : |\{ \alpha : x_\alpha \neq 0 \}| \leq \omega \}.
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**Theorem**

*If* \( C(K) \sim C(L) \) *where* \( L \) *is Corson compact* \n*then* \( K \) *has a \( \pi \) – base of sets having Corson compact closures. In particular, \( K \) *is itself Corson compact whenever* \( K \) *is homogeneous.*
Basic technique

If $\mu$ is a finite regular Borel measure on $K$, then $\mu$ is a continuous functional $C(K)$:

$$\mu(g) = \int g \, d\mu$$

for $\mu \in C(K)$. In fact, $C(K)^\ast$ can be identified with the space of all signed regular measures of finite variation (i.e. of the form $\mu_1 - \mu_2$, $\mu_1, \mu_2 - 0$).

Let $T : C(K) \to C(L)$ be a linear operator. Given $y \in L$, let $\delta_y \in C(L)^\ast$ be the Dirac measure. We can define $\nu_y \in C(K)^\ast$ by

$$\nu_y(g) = Tg(y)$$

for $g \in C(K)$. ($\nu_y = T^* \delta_y$).

Lemma: Let $T : C(K) \to C(L)$ be an embedding such that for $g \in C(K)$

$$m \cdot ||g|| \leq ||Tg|| \leq ||g||.$$

Then for every $x \in K$ and $m' < m$ there is $y \in L$ such that $\nu_y(\{x\}) > m'$. 
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G. Plebanek (IM UW)
Basic technique

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In fact, $C'(K)$ can be identified with the space of all signed regular measures of finite variation (i.e. is of the form $\mu_1 - \mu_2$, $\mu_1, \mu_2 \geq 0$).

Let $T : C(K) \to C(L)$ be a linear operator. Given $y \in L$, let $\delta_y \in C(L)'$ be the Dirac measure.

We can define $\nu_y \in C(K)'$ by $\nu_y(g) = Tg(y)$ for $g \in C(K)$ ($\nu_y = T^* \delta_y$).

Lemma

Let $T : C(K) \to C(L)$ be an embedding such that for $g \in C(K)$ $m \cdot \|g\| \leq \|Tg\| \leq \|g\|$. Then for every $x \in K$ and $m' < m$ there is $y \in L$ such that $\nu_y(\{x\}) > m'$.
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G. Plebanek (IM UWr)
An application

Theorem (W. Marciszewski, GP (2010))

Suppose that $C(K)$ embeds into $C(L)$, where $L$ is Corson compact. Then $K$ is Corson compact provided it has some measure-theoretic property (which holds true for all linearly ordered compacta and Rosenthal compacta).

Problem

Can one embed $C(2^{\omega_1})$ into $C(L)$, $L$ Corson?

No, if the embedding operator is to be positive or an isomorphism.

No, under MA+ non CH.

No, under CH (in fact whenever $2^{\omega_1} > c$).
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