Characterizing Strong Measure Zero Sets in Polish Groups

Galvin Mycielski Solovay Theorem Revisited

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Strong measure zero sets (in $\mathbb{R}$)

For an interval $I \subseteq \mathbb{R}$, let $\lambda(I)$ denote its length.

Definition (well-known)

A set $X \subseteq \mathbb{R}$ is (Lebesgue) measure zero ($X \in \mathcal{N}$) iff
for each positive real number $\varepsilon > 0$
there is a sequence of intervals $(I_n)_{n<\omega}$ of total length $\sum_{n<\omega} \lambda(I_n) \leq \varepsilon$
such that $X \subseteq \bigcup_{n<\omega} I_n$.

Definition (Borel; 1919)

A set $X \subseteq \mathbb{R}$ is strong measure zero ($X \in \mathcal{SN}$) iff
for each sequence of positive real numbers $(\varepsilon_n)_{n<\omega}$
there is a sequence of intervals $(I_n)_{n<\omega}$ with $\forall n \in \omega \;\; \lambda(I_n) \leq \varepsilon_n$
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Strong measure zero sets in Polish groups

Let $(G, +)$ be a (abelian?) Polish group.

Let $\mathcal{U}(0)$ denote the system of neighborhoods of the neutral element 0.

(Slightly?) abusing notation, I use the expression "strong measure zero" for subsets of a topological group.
Officially, the following property is called "Rothberger bounded":

Definition

$X \subseteq G$ is strong measure zero ($X \in \mathcal{SN}(G)$) if for every sequence $(U_n)_{n<\omega}$ of neighborhoods in $\mathcal{U}(0)$, there exists a sequence $(x_n)_{n<\omega}$ in G such that $X \subseteq \bigcup_{n<\omega}(x_n + U_n)$.

But note that this is the same as the official notion of "strong measure zero in metric spaces" for the metric space (G, d), where d is any (left-)invariant metric yielding the topology of G.

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Galvin Mycielski Solovay Theorem

Let $\mathcal{M}(G)$ be the (translation-invariant) σ-ideal of meager subsets of G.

For $X, M \subseteq G$, let $X + M = \{x + m : x \in X, m \in M\}$.

Definition

$X \subseteq G$ meager-shiftable ($X \in \mathcal{M}^*(G)$) if for every meager set $M \in \mathcal{M}(G)$, we have $X + M \neq G$.

Equivalently: $\forall M \in \mathcal{M}(G)\ \exists t \in G$ s.t. $(t + M) \cap X = \emptyset$.

Theorem (Galvin,Mycielski,Solovay; 1973)

A set $X \subseteq \mathbb{R}$ is strong measure zero if and only if for every meager set $M \in \mathcal{M}(\mathbb{R})$, $X + M \neq \mathbb{R}$, i.e.,

$$\mathcal{M}^*(\mathbb{R}) = \mathcal{SN}(\mathbb{R}).$$

The same holds for $(2^\omega, +)$, the 1-dimensial torus $(S^1, +) = (\mathbb{R}/\mathbb{Z}, +)$, ...

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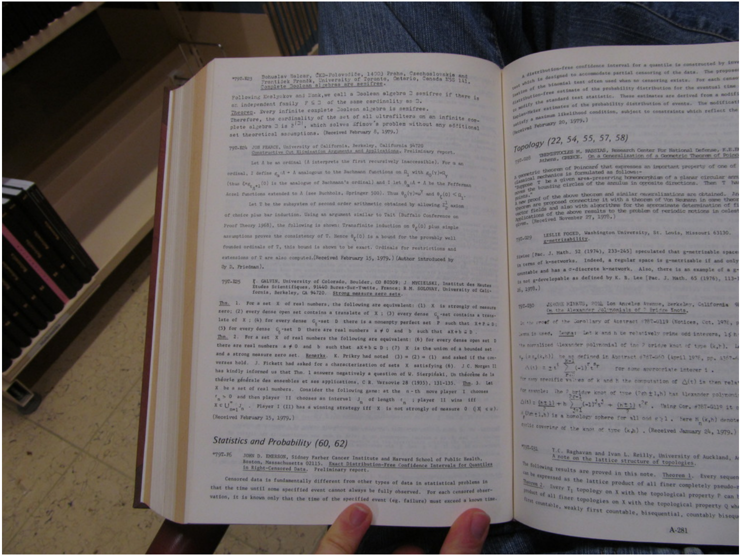

*79T-E23 Bohuslav Balcar, ČKD-Polovodiče, 14003 Praha, Czechoslovakia and Frantiček Franěk, University of Toronto, Ontario, Canada M5S 1A1. Complete Boolean algebras are semifree.

Following Koslyokov and Monk, we call a Boolean algebra B semifree if there is an independent family $F \subseteq B$ of the same cardinality as B. Theorem. Every infinite complete Boolean algebra is semifree. Therefore, the cardinality of the set of all ultrafilters on an infinite complete algebra B is $2^{|B|}$, which solves Efimov's problem without any additional set theoretical assumptions. (Received February 8, 1979.)

79T-E24 JON PEARCE, University of California, Berkeley, California 94720 Constructive Cut Elimination Arguments and Applications. Preliminary report.

Let Λ be an ordinal (Λ interprets the first recursively inaccessible). For α an ordinal, I define $\varepsilon_\alpha : \Lambda \to \Lambda$ analogous to the Bachmann functions on Ω_1 with $\varepsilon_0(\gamma) = \Omega_\gamma$ (thus $\xi = \varepsilon_{\varepsilon_\Lambda + 1}(0)$ is the analogue of Bachmann's ordinal) and I let $\theta_\alpha : \Lambda \to \Lambda$ be the Feferman Aczel functions extended to Λ (see Buchholz, Springer 500). Thus $\theta_0(\gamma) = \omega^\gamma$ and $\theta_\xi(0) < \Omega_1$.

Let T be the subsystem of second order arithmetic obtained by allowing Σ^1_2 axiom of choice plus bar induction. Using an argument similar to Tait (Buffalo Conference on Proof Theory 1968), the following is shown: Transfinite induction on $\theta_\xi(0)$ plus simple assumptions proves the consistency of T. Hence $\theta_\xi(0)$ is a bound for the provably well founded ordinals of T, this bound is shown to be exact. Ordinals for restrictions and extensions of T are also computed. (Received February 15, 1979.) (Author introduced by Sy D. Friedman).

79T-E25 F. GALVIN, University of Colorado, Boulder, CO 80309; J. MYCIELSKI, Institut des Hautes Études Scientifiques, 91440 Bures-Sur-Yvette, France; R. M. SOLOVAY, University of California, Berkeley, CA 94720. Strong measure zero sets.

Thm. 1. For a set X of real numbers, the following are equivalent: (1) X is strongly of measure zero; (2) every dense open set contains a translate of X; (3) every dense G_δ-set contains a translate of X; (4) for every dense G_δ-set D there is a nonempty perfect set P such that $X + P \subseteq D$; (5) for every dense G_δ-set D there are real numbers $a \neq 0$ and b such that $aX + b \subseteq D$. Thm. 2. For a set X of real numbers the following are equivalent: (6) for every dense open set D there are real numbers $a \neq 0$ and b such that $aX + b \subseteq D$; (7) X is the union of a bounded set and a strong measure zero set. Remarks. K. Prikry had noted (3) ⇒ (2) ⇒ (1) and asked if the converses hold. J. Fickett had asked for a characterization of sets X satisfying (6). J. C. Morgan II has kindly informed us that Thm. 1 answers negatively a question of W. Sierpiński, Un théorème de la théorie générale des ensembles et ses applications, C. R. Varsovie 28 (1935), 131-135. Thm. 3. Let X be a set of real numbers. Consider the following game: at the n-th move player I chooses $\varepsilon_n > 0$ and then player II chooses an interval J_n of length ε_n; player II wins iff $X \subseteq \bigcup_{n=1}^\infty J_n$. Player I (II) has a winning strategy iff X is not strongly of measure 0 ($|X| \leq \aleph_0$). (Received February 15, 1979.)

Statistics and Probability (60, 62)

*79T-F6 JOHN D. EMERSON, Sidney Farber Cancer Institute and Harvard School of Public Health, Boston, Massachusetts 02115. Exact Distribution-Free Confidence Intervals for Quantiles in Right-Censored Data. Preliminary report.

Censored data is fundamentally different from other types of data in statistical problems in that the time until some specified event cannot always be fully observed. For each censored observation, it is known only that the time of the specified event (eg. failure) must exceed a known time.

extensions of T are also computed.(Received February 15, 1979.) (Author introduced by Sy D. Friedman).

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Generalizing Galvin Mycielski Solovay Theorem?

The "easy direction" of the GMS theorem only uses separability:

Proposition

Let $(G, +)$ be a separable group. Then $\mathcal{M}^*(G) \subseteq \mathcal{SN}(G)$.

The "difficult direction" of the usual GMS theorem (for $\mathbb{R}, \dots$) makes essential use of the fact that the torus $\mathbb{R}/\mathbb{Z}$ is compact (and then "transfers" the result to $\mathbb{R}$).

Actually, compactness is already sufficient:

Theorem

Let $(G, +)$ be a compact group. Then $\mathcal{M}^*(G) \supseteq \mathcal{SN}(G)$.

Definition

A Polish group $(G, +)$ is a Galvin Mycielski Solovay group (GMS group) if the GMS theorem still holds, i.e., if ZFC proves that $\mathcal{M}^*(G) = \mathcal{SN}(G)$.

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Generalizing GMS theorem to (some?) σ-compact groups?

Corollary

Each compact Polish group $(G, +)$ is GMS, i.e., $\mathcal{M}^*(G) = \mathcal{SN}(G)$.

Definition

Let's say a Polish group $(G, +)$ is nicely σ-compact (different versions) if

- there exists a countable subgroup $U \subseteq G$ s.t. $(G/U, +)$ is compact
- there exists a selector $T \subseteq G$ for G/U s.t. either
  1. $\partial T (\cap T)$ is nowhere dense (meager?) in G
  2. $h[\partial T \cap T]$ is nowhere dense (meager?) as a subset of $(G/U, +)$, where $h : G \longrightarrow G/U$ is the canonical mapping.

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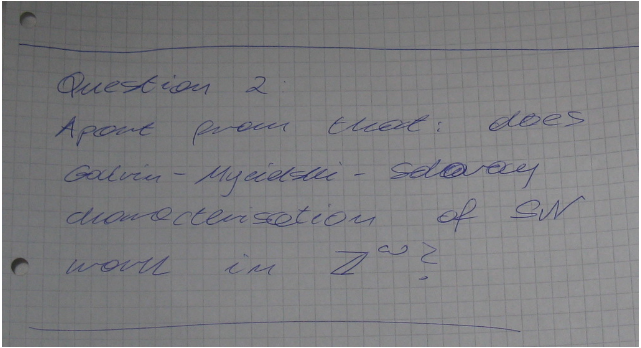
Theorem

Each nicely σ-compact Polish group $(G,+)$ is GMS $(\mathcal{M}^*(G) = \mathcal{SN}(G))$.

Marcin Kysiak's question

Marcin Kysiak asked me at the Winterschool 2011 here in Hejnice...



Question (Marcin Kysiak)

Is $(\mathbb{Z}^\omega, +)$ a GMS group, i.e., does $\mathcal{M}^*(\mathbb{Z}^\omega) = \mathcal{SN}(\mathbb{Z}^\omega)$ hold in general?

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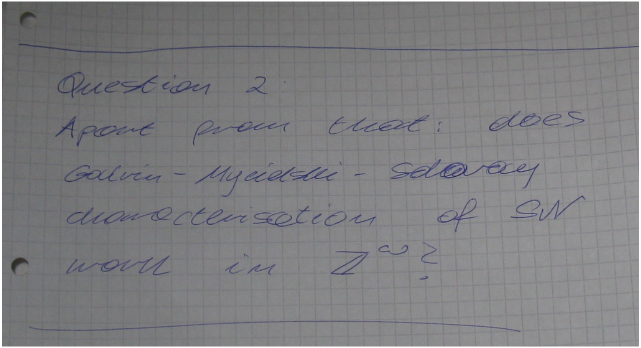

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$\mathbb{Z}^\omega$ is not a GMS group

Answer: No! (In other words: consistently, $\mathcal{M}^*(\mathbb{Z}^\omega) \neq \mathcal{SN}(\mathbb{Z}^\omega)$.)

Proposition

ZFC proves that $[\mathbb{Z}^\omega]^{\leq\aleph_0} \subseteq \mathcal{M}^*(\mathbb{Z}^\omega) \subseteq \mathcal{SN}(\mathbb{Z}^\omega)$.

It is quite easy to see that the usual BC (i.e., $\mathcal{SN}(2^\omega) = [2^\omega]^{\leq\aleph_0}$) is equivalent to the "Borel Conjecture on $\mathbb{Z}^\omega$" (i.e., $\mathcal{SN}(\mathbb{Z}^\omega) = [\mathbb{Z}^\omega]^{\leq\aleph_0}$).

Proposition

Assume BC. Then $[\mathbb{Z}^\omega]^{\leq\aleph_0} = \mathcal{M}^*(\mathbb{Z}^\omega) = \mathcal{SN}(\mathbb{Z}^\omega)$.

Theorem

Assume CH. Then $[\mathbb{Z}^\omega]^{\leq\aleph_0} \subsetneq \mathcal{M}^*(\mathbb{Z}^\omega) \subsetneq \mathcal{SN}(\mathbb{Z}^\omega)$.

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$\mathbb{Z}^\omega$ is not a GMS group

Answer: No! (In other words: consistently, $\mathcal{M}^*(\mathbb{Z}^\omega) \neq \mathcal{SN}(\mathbb{Z}^\omega)$.)

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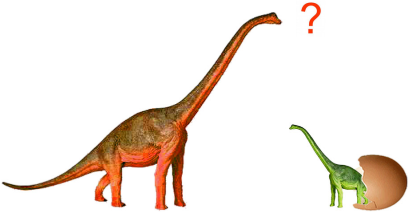

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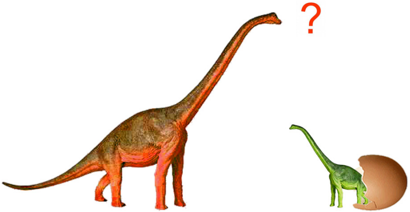

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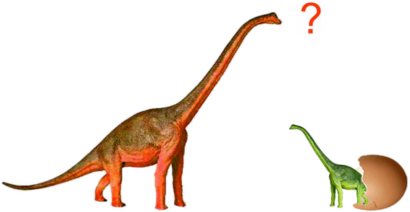

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Thank you for your attention and enjoy the Winter School. . .

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