Sequence selection principles for functions

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Sequence Selection Principles

A.V. Arkhangel’skii [1972]

properties $(\alpha_1) - (\alpha_4)$

For $i = 1, 2, 3, 4$, a topological space $Y$ is $(\alpha_i)$-space if for any sequence $\langle S_n : n \in \omega \rangle$ of sequences converging to a point $y \in Y$, there exists a sequence $S$ converging to $y$ such that:

$(\alpha_1)$ $S_n \setminus S$ is finite for all $n \in \omega$;
$(\alpha_2)$ $S_n \cap S$ is infinite for all $n \in \omega$;
$(\alpha_3)$ $S_n \cap S$ is infinite for infinitely many $n \in \omega$;
$(\alpha_4)$ $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.

D.H. Fremlin [1994]

equivalent conditions to an $s_1$-space

M. Scheepers [1997]

Sequence Selection Property SSP, Monotonic Sequence Selection Property MSSP

A topological space $X$ has sequence selection property, if for any $x \in X$ and for any sequence $\langle S_n : n \in \omega \rangle$ of sequences converging to $x$ there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n \to x$ and $x_n \in S_n$ for any $n \in \omega$. 
All spaces are assumed to be Hausdorff and infinite.

Diagrams hold for perfectly normal space.

- $X^\mathbb{R}$ the space of all real-valued functions on $X$ (Tychonoff topology = t. of pointwise convergence)
- $C_p(X)$ the space of all continuous functions on $X$ (subspace topology)
- $\mathcal{B}$ the space of all Borel functions on $X$ (subspace topology)
- $\mathcal{U}$ the space of all upper semicontinuous functions on $X$ with values in $[0,1]$ (subspace topology)

For a topological space $X$ the following are equivalent.

1. $X$ is an $s_1$-space.
2. $C_p(X)$ has the sequence selection property.
3. $C_p(X)$ possesses $(\alpha_2)$.
4. $C_p(X)$ possesses $(\alpha_3)$.
5. $C_p(X)$ possesses $(\alpha_4)$. 
perfectly normal space $X$

M. Scheepers [1998]  
L. Bukovský and J. Haleš [2007]  
M. Sakai [2007]  
B. Tsaban and L. Zdomskyy [2012]

$C_p(X)$ possesses $(\alpha_1)$.  
$X$ is a QN-space.  
$X$ is an $S_1(\Gamma_B, \Gamma_B)$-space.

M. Scheepers [1999]  
D.H. Fremlin [2003]  
L. Bukovský and J. Haleš [2007]  
M. Repický [2001]  
L. Bukovský and J. Haleš [2003]

$C_p(X)$ possesses $(\alpha_2)$.  
$X$ is a wQN-space.  
$X$ is an $S_1(\Gamma_{sh}, \Gamma)$-space.

$b$-Sierpiński set  
$X$ is a $\sigma$-set  
$X$ is zero-dimensional

$\gamma$-set  
$X$ is perfectly meager

compact set  
$X$ has count. Menger property
Convergence of \( \left\langle f_n : n \in \omega \right\rangle, \quad f_n, f : X \to \mathbb{R} \)

**Pointwise convergence P**

\[
f_n \xrightarrow{P} f
\]

\((\forall x \in X)(\forall \varepsilon > 0)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \rightarrow |f_n(x) - f(x)| < \varepsilon)\)

**Quasi-normal convergence Q**

\[
f_n \xrightarrow{Q} f
\]

there exists \( \langle \varepsilon_n : n \in \omega \rangle \) converging to 0 such that

\[(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \rightarrow |f_n(x) - f(x)| < \varepsilon_n)\]

**Discrete convergence D**

\[
f_n \xrightarrow{D} f
\]

\[(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \rightarrow f_n(x) = f(x))\]

**Monotonic convergence M**

\[
f_n \xrightarrow{M} f
\]

\[
f_n \xrightarrow{P} f \quad \text{and} \quad f_{n+1} \leq f_n \quad \text{for any} \quad n \in \omega
\]
Properties $AB(\mathcal{F},\mathcal{G})$ and $wAB(\mathcal{F},\mathcal{G})$

<table>
<thead>
<tr>
<th>$f_{0,0}$</th>
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<th>$f_{0,2}$</th>
<th>$f_{0,3}$</th>
<th>$\ldots$</th>
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$A, B \in \{P, Q, D\}$
pointwise P
quasi–normal Q
discrete D

$X$ has property $AB(\mathcal{F},\mathcal{G})$ if for any $f_{n,m} \in C_p(X)$, $f_n \in \mathcal{F}$, $f \in \mathcal{G}$ such that

$$f_{n,m} \overset{A}{\rightarrow} f_n \text{ for every } n \in \omega \text{ and } f_n \overset{A}{\rightarrow} f \text{ on } X$$

there exists an unbounded $\beta \in \omega \omega$ such that $f_{n,\beta(n)} \overset{B}{\rightarrow} f \text{ on } X$.

$X$ satisfies principle $wAB(\mathcal{F},\mathcal{G})$ if ... there exists an increasing $\alpha \in \omega \omega$ and an unbounded $\beta \in \omega \omega$ such that $f_{\alpha(n), \beta(n)} \overset{B}{\rightarrow} f \text{ on } X$. 

$\mathcal{F}, \mathcal{G} \subseteq X^\mathbb{R}$

$0 \in \mathcal{F}, \mathcal{G}$
**Sequence selection property** $\text{PP}(\{0\}, \{0\})$

was considered by A.V. Arkhangel’skiĭ [1972] as property $(\alpha_2)$ for $C_p(X)$ or M. Scheepers [1997] as sequence selection property for $C_p(X)$.

**Sequence selection property** $\text{wPP}(\{0\}, \{0\})$

was considered by A.V. Arkhangel’skiĭ [1972] as property $(\alpha_4)$ for $C_p(X)$.

**Sequence selection property** $\text{DP}(\{0\}, \{0\})$

was considered by L. Bukovský and J. Haleš [2007] as discrete sequence selection property.

**Sequence selection properties** $\text{AB}(X^R, X^R)$ and $\text{AB}(X^R, \{0\})$

were considered by L. Bukovský and J.Š. [2012] as ASB and ASB* selection principles.
\[ f_{n,m} \in C_p(X), \, n, m \in \omega, \quad f_{n,m} \xrightarrow{A} f_n \text{ for every } n \in \omega \text{ and } f_n \xrightarrow{A} f \text{ on } X \]

\[ f_n \text{ are } F_\sigma\text{-measurable functions on } X \]

\[ f \text{ is in second Baire class of functions on } X \]

We will use \( B \) instead of \( X^\mathbb{R} \).
If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$ then

$$AB(\mathcal{F}_2, \mathcal{G}_2) \to AB(\mathcal{F}_1, \mathcal{G}_1) \text{ and } wAB(\mathcal{F}_2, \mathcal{G}_2) \to wAB(\mathcal{F}_1, \mathcal{G}_1).$$

The family of sequence selection properties $AB(\mathcal{F}, \mathcal{G})$ and $wAB(\mathcal{F}, \mathcal{G})$ can be partially preordered by the relation

$$\mathcal{A} \leq \mathcal{D} \equiv \text{ZFC} \vdash \mathcal{D} \to \mathcal{A}.$$ 

Corresponding partially ordered set:

maximal elements are the equivalence classes of $PQ(\mathcal{B}, \mathcal{B})$ and $DD(\mathcal{B}, \mathcal{B})$
the smallest element is the equivalence class of $wDP(\{0\}, \{0\})$
maximal elements the equivalence classes of $PQ(B,B)$ and $DD(B,B)$
the smallest element the equivalence class of $wDP(\{0\},\{0\})$

L. Bukovský – J.Š. [2012]
A perfectly normal space $X$ has property $PQ(B,B)$ if and only if $X$ is a QN-space.
A perfectly normal space $X$ has property $DD(B,B)$ if and only if $X$ is a QN-space.

Corollary L. Bukovský, I. Reclaw and M. Repický [1991]
Any $b$-Sierpiński set has all selection properties $AB(\mathcal{F},\mathcal{G})$ and $wAB(\mathcal{F},\mathcal{G})$.

L. Bukovský – J. Haleš [2007], J.Š. [$\infty$]
A topological space $X$ has property $wDP(\{0\},\{0\})$ if and only if $X$ is a $wQN$-space.


$$AB(\mathcal{F},\mathcal{G}) \equiv MN(Q,H) \equiv wAB(\mathcal{F},\mathcal{G}) \equiv wMN(Q,H)$$
holds in Laver model.

A.W. Miller and B. Tsaban [2010]
In Laver model, a perfectly normal space $X$ has $AB(\mathcal{F},\mathcal{G})$ if and only if $|X| < b$.

L. Bukovský, I. Reclaw and M. Repický [1991]
A topological space $X$ is a QN-space (a $wQN$-space) if each sequence of continuous real-valued functions converging to zero on $X$ is (has a subsequence) converging quasi-normally.
A set $X \subseteq \mathbb{R}$ is $b$-Sierpiński set if $|X| \geq b$ and $|A \cap X| < b$ for any Lebesgue measure zero set.
\[ \text{AB}(\{0\}, \mathcal{G}) \equiv \text{AB}(\{0\}, \{0\}) \quad \text{wAB}(\{0\}, \mathcal{G}) \equiv \text{wAB}(\{0\}, \{0\}) \]

\[ \mathcal{G} \subseteq \mathcal{F}, \mathcal{G} \subseteq C_p(X), \mathcal{F} \text{ is closed under subtraction (L. Bukovský – J.Š. [2012])} \]

\[ \text{AB}(\mathcal{F}, \mathcal{G}) \equiv \text{AB}(\mathcal{F}, \{0\}) \quad \text{wAB}(\mathcal{F}, \mathcal{G}) \equiv \text{wAB}(\mathcal{F}, \{0\}) \]

\[ \text{wAB}(\{0\}, \{0\}) \equiv \text{AB}(\{0\}, \{0\}) \]

\[ (A, B) \neq (P, Q), \mathcal{F} \subseteq C_p(X) \]

\[ \text{AB}(\mathcal{F}, \mathcal{G}) \equiv \text{AB}(\mathcal{F}, \{0\}) \equiv \text{AB}(\{0\}, \{0\}) \equiv \text{wAB}(\mathcal{F}, \mathcal{G}) \equiv \text{wAB}(\mathcal{F}, \{0\}) \]
\( \mathcal{F}, \mathcal{G} \in \{\mathcal{B}, \mathcal{C}_p(X), \{0\}\} \)
A topological space $X$ is a $\gamma$-space if any open $\omega$-cover of $X$ contains $\gamma$-subcover.

M. Scheepers [1996]
A topological space $X$ is an $S_1(\Gamma, \Gamma)$-space if for every sequence $\langle A_n : n \in \omega \rangle$ of open $\gamma$-covers of $X$ there exist sets $U_n \in A_n, n \in \omega$ such that $\{U_n; n \in \omega\}$ is a $\gamma$-cover.

A topological space $X$ possesses $U_{fin}(\mathcal{O}_\omega, \Gamma)$ if for any sequence $\langle U_n : n \in \omega \rangle$ of countable open covers not containing a finite subcover there are finite sets $\mathcal{V}_n \subseteq U_n, n \in \omega$ such that $\bigcup \mathcal{V}_n; n \in \omega\}$ is a $\gamma$-cover.

For a property $\mathcal{A}$ of a topological space $X$ we say that $X$ is hereditarily $\mathcal{A}$-space, shortly $h\mathcal{A}$-space, or $X$ possesses $\mathcal{A}$ hereditarily if any subset of $X$ is an $\mathcal{A}$-space.

A topological space $X$ is a $\sigma$-set if every $F_\sigma$ subset of $X$ is a $G_\delta$ set in $X$. ($\leq 1933$)
A cover $\mathcal{A}$ of $X$ is an $\omega$-cover if for any finite subset $F$ of $X$ there is $A \in \mathcal{A}$ such that $F \subseteq A$.
An infinite cover $\mathcal{A}$ is a $\gamma$-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$. 

If $p = b$ then there is a $\gamma$-set of reals of cardinality $b$ which is not a $\sigma$-set.

A topological space $X$ is a $\gamma$-space if any open $\omega$-cover of $X$ contains $\gamma$-subcover.

A topological space $X$ is an $S_1(\Gamma, \Gamma)$-space if for every sequence $\langle A_n : n \in \omega \rangle$ of open $\gamma$-covers of $X$ there exist sets $U_n \in A_n$, $n \in \omega$ such that $\{U_n ; n \in \omega\}$ is a $\gamma$-cover.

A topological space $X$ possesses $U_{fin}(\mathcal{O}_\omega, \Gamma)$ if for any sequence $\langle U_n : n \in \omega \rangle$ of countable open covers not containing a finite subcover there are finite sets $\forall_n \subseteq U_n$, $n \in \omega$ such that $\{\bigcup \forall_n ; n \in \omega\}$ is a $\gamma$-cover.

For a property $A$ of a topological space $X$ we say that $X$ is hereditarily $A$-space, shortly h$A$-space, or $X$ possesses $A$ hereditarily if any subset of $X$ is an $A$-space.

A topological space $X$ is a $\sigma$-set if every $F_\sigma$ subset of $X$ is a $G_\delta$ set in $X$. ($\leq 1933$)

A cover $\mathcal{A}$ of $X$ is an $\omega$-cover if for any finite subset $F$ of $X$ there is $A \in \mathcal{A}$ such that $F \subseteq A$.

An infinite cover $\mathcal{A}$ is a $\gamma$-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$.
**Corollary**

\[ \text{Ind}(X) = 0 \] for any normal space \( X \) having any of the selection properties \( AB(\mathcal{F},\mathcal{G}) \) or \( wAB(\mathcal{F},\mathcal{G}) \). A subset of metric separable space having any of the selection properties \( AB(\mathcal{F},\mathcal{G}) \) or \( wAB(\mathcal{F},\mathcal{G}) \) is perfectly meager.

**J.Š. [∞]**

A perfectly normal space \( X \) having \( wDP(\mathcal{U},\{0\}) \) is an \( S_1(\Gamma,\Gamma) \)-space.

**L. Bukovský – J.Š. [2012]**

If a perfectly normal topological space \( X \) has \( wDP(\mathcal{U},\mathcal{B}) \) or \( DP(\mathcal{U},\{0\}) \) then \( X \) is a \( \sigma \)-set.

**Corollary**

If a perfectly normal space \( X \) has \( wDP(\mathcal{U},\mathcal{B}) \) or \( DP(\mathcal{U},\{0\}) \) then \( X \) is hereditarily \( S_1(\Gamma,\Gamma) \)-space.

**J.Š. [∞]**

If a perfectly normal topological space \( X \) has \( wDP(\mathcal{U},\{0\}) \) then every open \( \gamma \)-cover of \( X \) is shrinkable.

**J.Š. [∞]**

Let \( X \) be a topological space. If \( X \) has \( wDD(\{0\},\{0\}) \) or \( PQ(C_p(X),\{0\}) \) then \( X \) is a \( QN \)-space.

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A cover \( \mathcal{B} \) is said to be a refinement of \( \mathcal{A} \) if for any \( V \in \mathcal{B} \) there is \( U \in \mathcal{A} \) such that \( V \subseteq U \).

A \( \gamma \)-cover \( \mathcal{A} \) is shrinkable if there exists a closed \( \gamma \)-cover \( \mathcal{B} \) which is a refinement of \( \mathcal{A} \).
Surprising result

J.Š. $[\infty]$
Any $\gamma$-set has property $wPQ(\mathcal{B},\{0\})$.

\[
AB\{0\},\{0\} \equiv wAB\{0\},\{0\}
\]
\[
AB(C_p(X),\mathcal{B}) \equiv wAB(C_p(X),\mathcal{B})
\]
for $A, B \neq P, Q : AB(C_p(X),\{0\}) \equiv wAB(C_p(X),\{0\})$

p = b

\[
\begin{align*}
\text{QN} & \rightarrow \text{PP}(\mathcal{B},\{0\}) & \rightarrow \text{DP}(\mathcal{B},\{0\}) & \rightarrow \text{h}S_1(\Gamma, \Gamma) & \rightarrow \sigma\text{-set} \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow \\
\gamma & \rightarrow wPQ(\mathcal{B},\{0\}) & \rightarrow wPP(\mathcal{B},\{0\}) & \rightarrow wDP(\mathcal{B},\{0\}) & \rightarrow S_1(\Gamma, \Gamma) & \rightarrow wQN
\end{align*}
\]
Distinguishing

\[ p = b, \quad (A, B) \neq (P, Q), \quad B \neq D \]

\[ QN \equiv PQ(C_p(X),\{0\}) \rightarrow AB(\mathcal{B},\mathcal{B}) \]

\[ \text{wAB(\mathcal{B},\mathcal{B})} \]

\[ \text{hS}_1(\Gamma, \Gamma) \rightarrow \sigma\text{-set} \]

\[ AB(\mathcal{B},\{0\}) \]

\[ \text{wQN} \equiv \text{wPQ}(C_p(X),\{0\}) \equiv AB(C_p(X),\{0\}) \equiv AB(\{0\},\{0\}) \equiv PQ(\{0\},\{0\}) \]

Miller model, \( (A, B) \neq (P, Q), \ B \neq D \)
A.W. Miller [1979]

\[ QN \equiv PQ(C_p(X),\{0\}) \equiv AB(\mathcal{B},\mathcal{B}) \equiv \text{wAB(\mathcal{B},\mathcal{B})} \equiv AB(\mathcal{B},\{0\}) \equiv \text{hS}_1(\Gamma, \Gamma) \equiv \sigma\text{-set} \]

\[ \text{wQN} \equiv \text{wPQ}(C_p(X),\{0\}) \equiv AB(C_p(X),\{0\}) \equiv AB(\{0\},\{0\}) \equiv PQ(\{0\},\{0\}) \]
perfectly normal space $X$

<table>
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<th>$X$ is a QN-space</th>
<th>$X$ is hereditarily $S_1(\Gamma, \Gamma)$-space</th>
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L. Bukovský - J.Š. [2012]  
J.Š. [∞]  
J.Š. [∞]  
M. Scheepers [1999], D.H. Fremlin [2003]

Properties $\text{ABC}(\mathcal{F},\mathcal{G})$ and $\text{wABC}(\mathcal{F},\mathcal{G})$

$\begin{array}{cccccccc}
 f_{0,0} & f_{0,1} & f_{0,2} & f_{0,3} & \ldots & f_{0,m} & \ldots & \xrightarrow{A} f_0 \\
 f_{1,0} & f_{1,1} & f_{1,2} & f_{1,3} & \ldots & f_{1,m} & \ldots & \xrightarrow{A} f_1 \\
 f_{2,0} & f_{2,1} & f_{2,2} & f_{2,3} & \ldots & f_{2,m} & \ldots & \xrightarrow{A} f_2 \\
 \vdots & \vdots & \vdots & \vdots & \ & \ & \ & \ \\
 f_{n,0} & f_{n,1} & f_{n,2} & f_{n,3} & \ldots & f_{n,m} & \ldots & \xrightarrow{A} f_n \\
 \vdots & \vdots & \vdots & \vdots & \ & \ & \ & \ \\
 \end{array}$

$\begin{array}{cccc}
 f_0 & f_1 & f_2 & \ldots \\
 f_{0,m} & f_{1,m} & f_{2,m} & \ldots \\
 \vdots & \vdots & \vdots & \vdots \\
 f_{n,m} & \ldots & \ & \ \\
 \end{array}$

$\xrightarrow{\mathcal{A}}$ $\xrightarrow{\mathcal{B}}$ $\xrightarrow{\mathcal{C}}$

$A, B, C \in \{P, Q, D, M\}$

pointwise P

quasi–normal Q

discrete D

monotonic M

$\mathcal{F}, \mathcal{G} \subseteq X^\mathbb{R}$

$0 \in \mathcal{F}, \mathcal{G}$

$X$ has property $\text{ABC}(\mathcal{F},\mathcal{G})$ if for any $f_{n,m} \in C_p(X)$, $f_n \in \mathcal{F}$, $f \in \mathcal{G}$ such that

$f_{n,m} \xrightarrow{A} f_n$ for every $n \in \omega$ and $f_n \xrightarrow{B} f$ on $X$

there exists an unbounded $\beta \in \omega^\omega$ such that $f_{n,\beta(n)} \xrightarrow{C} f$ on $X$.

$X$ satisfies principle $\text{wABC}(A,B)$ if \ldots there exists an increasing $\alpha \in \omega^\omega$ and an unbounded $\beta \in \omega^\omega$ such that $f_{\alpha(n),\beta(n)} \xrightarrow{C} f$ on $X$. 
perfectly normal space $X$

$X$ is a QN-space $X$ has $PQ(\mathcal{B},\mathcal{B})$ L. Bukovský - J.Š. [2012]

$X$ is hereditarily $S_1(\Gamma, \Gamma)$-space $X$ has $PP(\mathcal{U}, \{0\})$ J.Š. [$\infty$]

$X$ is an $S_1(\Gamma, \Gamma)$-space and every open $\gamma$-cover of $X$ is shrinkable $X$ has $wPQ(\mathcal{U}, \{0\})$ J.Š. [$\infty$]

$X$ is a $wQN$-space $X$ has $PP(\{0\},\{0\})$ M. Scheepers [1999], D.H. Fremlin [2003]

$X$ possesses Hurewicz property hereditarily $X$ has $MPP(\mathcal{B},\{0\})$ J.Š. [$\infty$]

$X$ possesses Hurewicz property and every open $\gamma$-cover of $X$ is shrinkable $X$ has $wMPP(\mathcal{B},\{0\})$ J.Š. [$\infty$]

$X$ possesses $USC_m$ and Hurewicz property $X$ has $MMP(\mathcal{B},\{0\})$ J.Š. [$\infty$]

$X$ possesses Hurewicz property $X$ has $MMP(\{0\},\{0\})$ M. Scheepers [1997]


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Hurewicz property = property $U_{fin}(\mathcal{O}_\omega, \Gamma)$

Property $USC_m$ introduced and investigated by H. Ohta and M. Sakai [2009].
M. Scheepers [1999], D.H. Fremlin [2003] \( B \neq D \)

\( \top \)

\( X \) satisfies \( AB(\{0\},\{0\}) \) if and only if \( X \) is a wQN-space.

\( \sp{\top} \)

\( X \)

J.Š. \([\infty]\) \( X \) is a wQN-space if and only if \( X \) has \( wPQ(C_p(X),\{0\}) \).

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L. Bukovský – J.Š. [2012] \( X \) has \( DD(\mathcal{F},\mathcal{G}) \) if and only if \( X \) is a QN-space.

\( \p{n} \)

\( X \) has \( wDD(\mathcal{F},\mathcal{G}) \) if and only if \( X \) is a QN-space.

\( \sp{\p{n}} \)

\( X \)

L. Bukovský – J.Š. [2012], J.Š. \([\infty]\) \( C_p(X) \subseteq \mathcal{F} \) \( X \) has \( PQ(\mathcal{F},\mathcal{G}) \) if and only if \( X \) is a QN-space.

\( X \) has \( wPQ(\mathcal{F},\mathcal{B}) \) if and only if \( X \) is a QN-space.

L. Bukovský – J.Š. [2012] \( X \) is a QN-space if and only if \( X \) has \( wQQ(\mathcal{B},\mathcal{B}) \) if and only if \( X \) has \( QQ(\mathcal{B},\mathcal{B}) \).

L. Bukovský – J.Š. \([\infty]\) \( QQ(\mathcal{B},\{0\}) \equiv DQ(\mathcal{B},\{0\}) \equiv QP(\mathcal{B},\{0\}) \equiv DP(\mathcal{B},\{0\}) \)

\( wQQ(\mathcal{B},\{0\}) \equiv wDQ(\mathcal{B},\{0\}) \equiv wQP(\mathcal{B},\{0\}) \equiv wDP(\mathcal{B},\{0\}) \)

J.Š. \([\infty]\) \( B \neq D \)

If \( (A, B) \neq (P, Q) \) then \( X \) has \( AB(U,\{0\}) \) if and only if \( X \) is hereditarily \( S_1(\Gamma, \Gamma) \)-space.

\( X \) has \( wAB(U,\{0\}) \) if and only if \( X \) is an \( S_1(\Gamma, \Gamma) \)-space and every open \( \gamma \)-cover of \( X \) is shrinkable.
Application

1) some principles can be described by sequential closure operator in $X^\mathcal{R}$

2) an alternative proof of Tsaban–Zdomskyy Theorem

3) an alternative proof of strengthened Reclaw Theorem
Application

1) some principles can be described by sequential closure operator in $\mathbb{X}$

2) an alternative proof of Tsaban–Zdomskyy Theorem

3) an alternative proof of strengthened Reclaw Theorem
Let $X$ be a topological space.

**D.H. Fremlin [1994], M. Scheepers [1999]**

$X$ has $\text{PP}\{0\}/\text{0}$ if and only if $\text{scl}_{\omega_1}(A, C_p(X)) = \text{scl}_1(A, C_p(X))$ for every $A \subseteq C_p(X)$.

**J.Š. [∞]**

$X$ has $\text{wPP}(\mathcal{B},\mathcal{B})$ if and only if $\text{scl}_{\omega_1}(A, X^\mathbb{R}) = \text{scl}_1(A, X^\mathbb{R})$ for every $A \subseteq C_p(X)$.

$X$ has $\text{wPP}(\mathcal{B},\{0\})$ if and only if $\text{scl}_2(A, X^\mathbb{R}) \cap C_p(X) = \text{scl}_1(A, X^\mathbb{R}) \cap C_p(X)$ for any $A \subseteq C_p(X)$.

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$A \subseteq Y$: $\text{scl}(A, Y) = \{y \in Y; \exists \{y_n\}_{n=0}^\infty \in \omega A \ y_n \to y\}$

$\text{scl}_0(A, Y) = A$, $\text{scl}_\alpha(A, Y) = \text{scl} \left( \bigcup_{\beta < \alpha} \text{scl}_\beta(A, Y), Y \right), \alpha > 0$
T. Orenshtein [2009]

$X$ possesses property $(S'_0)_{\Gamma_0}$ if for any set $A \subseteq C_p(X) \setminus \{0\}$ with $0 \in \text{scl}_{\omega_1}(A, X)$ there is a sequence $\langle f_n : n \in \omega \rangle$ of functions from $A$ such that $f_n \to 0$.

J.Š. [∞]

The statements

"$\text{scl}_{\omega_1}(A, X) = \text{scl}_1(A, X)$ for every $A \subseteq C_p(X)$ for any perfectly normal $S_1(\Gamma, \Gamma)$-space $X$",

"$\text{scl}_{\omega_1}(A, X) = \text{scl}_1(A, X)$ for every $A \subseteq C_p(X)$ for any perfectly normal space $X$ possessing $(S'_0)_{\Gamma_0}$"

are undecidable in ZFC. The theory

$\text{ZFC} + \text{“any perfectly normal $S_1(\Gamma, \Gamma)$-space possesses $(S'_0)_{\Gamma_0}$”}$

is consistent with ZFC.

Solutions and partial solution to Problems 6.0.15, 6.0.16 and 6.0.17 of T. Orenshtein [2009].

\[
A \subseteq Y: \text{scl}(A, Y) = \{ y \in Y; (\exists \{ y_n \}_{n=0}^{\infty} \in \omega A) y_n \to y \}
\]

\[
\text{scl}_0(A, Y) = A, \text{scl}_\alpha(A, Y) = \text{scl} \left( \bigcup_{\beta < \alpha} \text{scl}_\beta(A, Y), Y \right), \alpha > 0
\]
Application

1) some principles can be described by sequential closure operator in $\mathcal{X}^\mathcal{R}$

2) an alternative proof of Tsaban–Zdomskyy Theorem

3) an alternative proof of strengthened Reclaw Theorem
B. Tsaban – L. Zdomskyy [2012], announcement 2006
If $X$ is a perfectly normal topological space, then $X$ is a QN-space if and only if any Borel measurable function $f : X \to \omega\omega$ is eventually bounded.

L. Bukovský – J.Š. [∞]
A topological space $X$ possesses the JR-property if every $\Delta^0_2$ measurable real function defined on $X$ is a discrete limit of a sequence of continuous functions.

J.E. Jayne and C.A. Rogers 1982 Any analytic subset of a Polish space has the JR-property.

L. Bukovský, I. Reclaw and M. Repický [2001]
If $X$ is a perfectly normal topological space, then $X$ is a QN-space with the JR-property if and only if any Borel measurable function $f : X \to \omega\omega$ is eventually bounded.

L. Bukovský – J.Š. [2012]
Any QN-space has property QQ($\mathcal{B},\mathcal{B}$).

L. Bukovský – J.Š. [2012]
If a perfectly normal space $X$ has QQ($\mathcal{B},\mathcal{B}$), then $X$ has the JR-property.
Application

1) some principles can be described by sequential closure operator in $\mathbb{X}\mathbb{R}$

2) an alternative proof of Tsaban–Zdomskyy Theorem

3) an alternative proof of strengthened Reclaw Theorem
A perfectly normal QN-space is a $\sigma$-space.

Any QN-space has property $QQ(B,B)$.

If a perfectly normal topological space $X$ has $wDP(U,B)$ then $X$ is a $\sigma$-set.


Galvin F. and Miller A.W., $\gamma$-sets and other singular sets of real numbers, Topology Appl. 17 (1984), 145–155.


Scheepers M., *Sequential convergence in $C_p(X)$ and a covering property*, East-West J. of Mathematics **1** (1999), 207–214.


Thanks for Your attention!