Generating Borel Functions with Continuous Functions

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joint work with

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Let S be a semigroup and let $U \subseteq S$.

Definition

The relative rank of S with respect to U is the minimal cardinality of a subset $V \subseteq S$ such that $\langle U \cup V \rangle = S$, that is, U together with V generate S. We denote the relative rank by $r(S : U)$.

Theorem (Sierpiński, 1935)

The relative rank of the semigroup of all mappings from an infinite set A to A with respect to any subsemigroup is either uncountable or finite and then equal to 0, 1 *or* 2.

Let X be a linearly ordered set. Let $\mathcal{O}_X$ be the set of all order-preserving functions from X to X.

Theorem (Higgins, Howie, Mitchell, Ruškuc, 2003)

Let X be a countably infinite linearly ordered set, or an infinite well-ordered set (of arbitrary cardinality). Then the relative rank of ${}^X X$ with respect to $\mathcal{O}_X$ is 1.

Introduction

If X is a metric space then let $L(X)$ and $C(X)$ be the class of all Lipschitz and all continuous functions from X to X, respectively. Let $\mathcal{N} = {}^{\mathbb{N}}\mathbb{N}$ be a metric space with the metric $d(x, y) = 1/n$ where $\mathbb{N} = \{1, 2, \ldots\}$ and n is the first coordinate such that $x_n \neq y_n$, for $x \neq y$.

Theorem (Cichoń, Mitchell, Morayne, 2007)

If $\mathcal{N}$ is defined as above then we have $r(C(\mathcal{N}) : L(\mathcal{N})) = \aleph_1$. If $x = (1, 1, \ldots)$ then $r(C(\mathcal{N} \setminus \{x\}) : L(\mathcal{N} \setminus \{x\})) = 1$.

Main result

Let $B(X)$ be the family of all Borel functions from X to X.

Theorem

If an uncountable Polish space X satisfies one of the following conditions

- X *is* 0*-dimensional,*
- X *is homeomorphic to its square,*
- X *contains a homeomorphic copy of the interval* $[0, 1]$*,*

then $r(B(X) : C(X)) = \aleph_1$.

Proof of the inequality $r(B(X) : C(X)) \geqslant \aleph_1$

Suppose that $r(B(X) : C(X)) \leqslant \aleph_0$.
Then there is a family $\{\psi_n : n < \omega\} \subseteq B(X)$ such that $\langle C(X) \cup \{\psi_n : n < \omega\}\rangle = B(X)$.
Let
$B_\alpha(X, Y) = \{f \in {}^X Y : f^{-1}[U] \in \Sigma^0_{1+\alpha}(X) \text{ for each } U \in \Sigma^0_1(Y)\}$
and $B_\alpha(X) = B_\alpha(X, X)$.
We have:

Fact

For every $f \in B(X)$ there is an $\alpha < \aleph_1$ such that $f \in B_\alpha(X)$.

Thus for every $n < \omega$ there is an $\alpha_n < \aleph_1$ such that $\psi_n \in B_{\alpha_n}(X)$.
Let $\gamma = \sup_{n<\omega} \alpha_n + \omega$.

Proof of the inequality $r(B(X) : C(X)) \geqslant \aleph_1$

Knowing that $g \circ f \in B_{\alpha+1+\beta+1}(X)$ for $f \in B_\alpha(X)$ and $g \in B_\beta(X)$, we have that

$$f_k \circ \psi_{n_{k-1}} \circ \ldots \circ f_1 \circ \psi_{n_0} \circ f_0 \in B_{\gamma\omega}(X)$$

for any $f_0, \ldots, f_k \in C(X)$ and $n_0, \ldots, n_{k-1} < \omega$.
This leads to a contradiction, since $B_{\gamma\omega}(X) \subsetneq B(X)$.

Preparation for a proof of $r(B(X) : C(X)) \leqslant \aleph_1$

Since X is uncountable there are Cantor sets $D, E \subseteq X$ and homeomorphism $\phi : D \times D \to E$.

Fact

Every nonempty and closed subset of a 0-dimensional metric space is its retract.

Thus if X is 0-dimensional then every continuous function $f \in C(D, E)$ has an extension $g \in C(X, E)$.
Then

$$(*) \text{ for every } d \in D \text{ there is an } f \in C(X) \text{ such that } f|D = \phi(d, \cdot).$$

If X contains a homeomorphic copy I of the unit interval, then adding requirement $E \subseteq I$ we can use the Tietze extension theorem to make X satisfy the condition $(*)$.

If X is homeomorphic to its square, then there is a continuous injection $h : D \times X \to X$.
In this case we define $\phi = h|(D \times D)$.
Then $\phi(d, \cdot) = h(d, \cdot)|D$ and $h(d, \cdot) \in C(X)$ for any $d \in D$.
Thus in this case X also satisfies the condition $(*)$.

Lemma

Assume that there are Cantor sets D, E contained in an uncountable Polish space X which satisfy the condition $(*)$, i.e. for every $d \in D$ there is an $f \in C(X)$ such that $f|D = \phi(d, \cdot)$. Then for every Borel function $F : D \times X \to X$ there are Borel functions $G, H \in B(X)$ such that for any $d \in D$ there is an $f \in C(X)$ such that $F(a, \cdot) = G \circ f \circ H$.

Let us recall that (Kuratowski, 1934) if X, Y are Polish spaces of the same cardinality then there exists a bijection $f \in B_1(X, Y)$ from X onto Y such that $f^{-1} \in B_1(Y, X)$.
Thus there is a bijection $H : X \to D$ such that $H \in B_1(X, D)$ and $H^{-1} \in B_1(D, X)$.
For every $e \in E$ we define

$$G(e) = F(\pi_1(\phi^{-1}(e)), H^{-1}(\pi_2(\phi^{-1}(e)))),$$

where π_1, π_2 are projections.
We see that $G(\phi(a, b)) = F(a, H^{-1}(b))$ for every $a, b \in D$.
Fix $d \in D$.
From $(*)$ there is an $f \in C(X)$ such that $\phi(d, \cdot) = f|D$.
Thus for every $x \in X$,

$$G(f(H(x))) = G(\phi(d, H(x))) = F(d, H^{-1}(H(x))) = F(d, x).$$

Proof of the inequality $r(B(X) : C(X)) \leqslant \aleph_1$

Fact

Let D be a Cantor set. For each $\alpha < \aleph_1$ there is a Borel function $F_\alpha : D \times X \to X$ which is universal for the class $B_\alpha(X)$, i.e. for any $f \in B_\alpha(X)$ there is a $d \in D$ such that $f = F_\alpha(d, \cdot)$.

From the previous lemma there are functions $G_\alpha, H_\alpha \in B(X)$ such that for every $d \in D$ there is an $f \in C(X)$ such that
$F_\alpha(d, \cdot) = G_\alpha \circ f \circ H_\alpha$.
Now it suffices to show that

$$\langle C(X) \cup \{G_\alpha : \alpha < \aleph_1\} \cup \{H_\alpha : \alpha < \aleph_1\}\rangle = B(X).$$

Fact

Let D be a Cantor set. For each $\alpha < \aleph_1$ there is a Borel function $F_\alpha : D \times X \to X$ which is universal for the class $B_\alpha(X)$, i.e. for any $f \in B_\alpha(X)$ there is a $d \in D$ such that $f = F_\alpha(d, \cdot)$.

Fix $g \in B(X)$.
Then from the fact that $B(X) = \bigcup_{\alpha<\aleph_1} B_\alpha(X)$ there is an $\alpha < \aleph_1$ such that $g \in B_\alpha(X)$.
There is also a $d \in D$ such that $F_\alpha(d, \cdot) = g$.
Functions G_α, H_α were chosen in such a way that there is an $f \in C(X)$ such that $F_\alpha(d, \cdot) = G_\alpha \circ f \circ H_\alpha$.
Thus

$$g = G_\alpha \circ f \circ H_\alpha \in \langle C(X) \cup \{G_\beta : \beta < \aleph_1\} \cup \{H_\beta : \beta < \aleph_1\}\rangle.$$

Question

Is there an uncountable Polish space X such that $\aleph_1 < r(B(X) : C(X)) < \mathfrak{c}$?