Generalization of Strok-Szymański theorem

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Winter School, Hejnice, 26-01-2013 - 2-02-2013

Motivation

Space X is supercompact if exists a subbase s.t for every open covering of X consisting of subbasic sets there is a 2-element subcover.

Alexander lemma implies that every supercompact space is compact

Definition

A family of sets $\mathcal{L}$ is called linked if any two members of this family have nonempty intersection.

A family of sets $\mathcal{L}$ is called binary if each linked subfamily of $\mathcal{L}$ has nonempty intersection.

Lemma

A space X is supercompact iff it possesses a binary subbbase for closed sets.

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Theorem (Strok, Szymański 1975)

Every metric compact space is supercompact.

Definition

Let $\mathcal{P}$ be a collection of subsets of a topological space X.
$\mathcal{P}$ is called *k-network* if for any compact subset K of space X and its open neighbourhood U exists a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{P} \subset U$.

Definition

A space X is called $\aleph$-space if it posseses a σ-locally finite k-network.

Theorem (Foged 1984)

The following are equivalent for a regular space X:

1. *X has a σ-locally finite k-network,*
2. *X has a σ-discrete k-network.*

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Question

Does every $\aleph$-space possess a σ-discrete binary (in finite sense) k-network for closed sets?

Basic facts

Lemma

For every family $\mathcal{B}$ of finite order in X exists an essential map $\lambda\colon X \to K$ onto a finite dimensional complex K such that

$$\lambda(\bigcap_{i=1}^{n} B_i) = \bigcap_{i=1}^{n} \lambda(B_i)$$

for all $n \in \omega$ and for all $B_1, \ldots, B_n \in \mathcal{B}$.

Lemma

For every finite dimensional complex K and any finite family $\mathcal{A}$ of subcomplexes of K and any linked finite non-empty family $\mathcal{B}$ of closed stars of the second barycentric subdivision we have

$$(\forall A \in \mathcal{A})(\forall B \in \mathcal{B})(A \cap B \neq \emptyset \Rightarrow \bigcap \mathcal{A} \cap \bigcap \mathcal{B} \neq \emptyset$$

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Lemma

If $f\,X \to Y$ is a map onto Y and $\mathcal{B}$ is a binary collection in Y, then $f^{-1}(\mathcal{B}) = \{f^{-1}(Z) \colon Z \in \mathcal{B}\}$ is a binary collection.

Main result

Theorem

Every normally $\aleph$-space possesses σ-discrete, binary, closed k-network.