Hereditarily supercompact spaces

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Winter School, Hejnice 2013

Supercompactness

A topological space X is supercompact if it has a subbase of the topology such that each cover of X by elements of this subbase has a two element subcover (J. de Groot, 1967).
The Alexander Lemma $\Rightarrow$ supercompact space is compact.

Example 1 (of supercompact spaces)

1. linearly ordered compact spaces;
2. compact metrizable spaces (M. Strok, A. Szymanski; 1975);
3. compact topological groups (C. Mills 1978).

$\beta\omega$ is not supercompact . A supercompact space has non-trival convergent sequence (E. van Douwen, J. van Mill 1982).

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Supercompactness

1. Product of supercompact spaces is supercompact (Cantor cube $\{0,1\}^\kappa$ and Tichonoff cube $[0,1]^\kappa$ are supercompact).
2. One-point compactification of topological sum of supercompact spaces is supercompact (Aleksandroff compactification of discrete space αX is supercompact).

A continuous image of a supercompact space need not to be supercompact (J. van Mill & C.F. Mills 1979). There are dyadic spaces (=continuous image of Cantor cube $\{0;1\}^\kappa$) that are not supercompact (M. Bell 1990).

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Hereditarily supercompact spaces

A topological space X is called hereditarily supercompact if each closed subspace of X is supercompact.

Which topological spaces are hereditarily supercompact?

Strok, Szymański result implies that: each compact metric space is hereditarily supercompact.

Theorem 1

A compact topological space X is hereditarily supercompact if X contains a subspace $Z \subseteq X$ such that

1. *Z is hereditarily supercompact;*
2. *$X \setminus Z$ is discrete.*
3. *Z is a retract of X;*

Corollary 1

The Aleksandroff duplicate $A(X)$ of hereditarily supercompact space X is hereditarily supercompact.

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Since there exists a compact and not supercompact space of weight ω_1, and each non-metrizable dyadic compact space X contains a topological copy of the Cantor cube $\{0,1\}^{\omega_1}$ (Gerlits, Efimov) then:

Proposition 1

A dyadic compact space is hereditarily supercompact if and only if it is metrizable.

More non-trival examples: monotonically normal spaces

Theorem 2

Each monotonically normal compact space is hereditarily supercompact.

Theorem (W. Bula, J. Nikiel, M. Tuncali, E. Tymchatyn)

Each continuous image of a linearly ordered compact space is supercompact.

Theorem (M. E. Rudin)

A compact space is monotonically normal if and only if it is a continuous image of a linearly ordered compact space.

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Theorem (A. J. Ostaszewski 1978)

Each monotonically normal space X is sub-hereditarily separable space i.e. each separable subspace of X is hereditarily separable.

monotonically normal spaces $\subseteq$ sub-hereditarily separable spaces

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Theorem 3 (T.Banakh, Z. Kosztolowicz, S.T.)

Under $\omega_1 < \mathfrak{p}$, each hereditarily supercompact space is sub-hereditarily separable.

$\mathfrak{p}$ is the smallest cardinality of a base of a free filter $\mathcal{F}$ on a countable set X, which has no infinite pseudo-intersection.

Corollary 2 (MA+¬ CH)

Each separable hereditarily supercompact space is hereditarily separable and hereditarily Lindelöf.

Scattered space

Theorem (T. Banakh, A. Leiderman 2012)

The class of scattered compact hereditarily paracompact spaces equals to the smallest class $\mathcal{A}$ which contains the singleton and is closed with respect to taking the one-point compactification of a topological sum $\bigoplus_{i \in I} X_i$ of spaces from the class $\mathcal{A}$.

Proposition 2

For any hereditarily supercompact spaces X_i, $i \in I$, the one-point compactification αX of the topological sum $X = \bigoplus_{i \in I} X_i$ is hereditarily supercompact. In particular, the one-point compactification αD of any discrete space D is hereditarily supercompact.

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Theorem 4

Each scattered compact hereditarily paracompact space is hereditarily supercompact.

Theorem (T. Banakh, A. Leiderman 2012)

A scattered compact space is metrizable if and only if it is separable and hereditarily paracompact.

The preceding two theorems motivate the following problem.

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Is each separable scattered hereditarily supercompact space metrizable?

Yes (under MA + $\neg$ CH)

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$\{0,1\}^\kappa$ and $[0,1]^\kappa$ are not hereditarily supercompact, $\kappa > \omega_0$, since there is a compact zero-dimensional space of weight ω_1, which is not supercompact.

Is the product of finitely many hereditarily supercompact spaces hereditarily supercompact?

Theorem 5 (T.Banakh, Z. Kosztolowicz,S.T)

Let $X \subseteq [0,1] \times \alpha D$ be a closed subset of the product of the closed unit interval $[0,1]$ and the one-point compactification $\alpha D = \{\infty\} \cup D$ of a discrete space D. If the space X is supercompact, then the subspace $\mathcal{X} = \{X_i : i \in D\}$ is meager in the hyperspace $\exp([0,1])$, where $X_i = \{x \in [0,1] \colon (x,i) \in X\}$ is the i-th section of the set X in the product $[0,1] \times \alpha D$, $i \in D$.

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$\mathcal{F} = \{X_d \colon d \in D\} \subseteq \exp([0,1])$, $\mathcal{F}$ is non-meager in $\exp([0,1])$ and $|\mathcal{F}| = |D| = \mathrm{non}(\mathcal{M})$, where $\mathcal{M}$ is the ideal of meager sets in $\exp([0,1])$

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