

Isometric uniqueness of a complementably universal Banach space for Schauder decomposition.

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We present an isometric version of the complementably universal Banach space \mathbb{P} with a Schauder decomposition. The space \mathbb{P} is isomorphic to Pełczyński's space with a universal basis as well as to Kadec's complementably universal space with the bounded approximation property.

In 1969 Pełczyński constructed a complementably universal Banach space with a Schauder basis. Two years later, Kadec constructed a complementably universal Banach space for the class of spaces with the BAP. Just after, Pełczyński showed that every Banach space with BAP is complemented in a space with a basis. Applying Pełczyński' decomposition argument, one immediately concludes that both spaces are isomorphic.

- Let X, Y be Banach spaces, $\varepsilon > 0$. $f : X \rightarrow Y$ is an ε -isometry if

$$(1 + \varepsilon)^{-1} \cdot \|x\| \leq \|f(x)\| \leq (1 + \varepsilon) \cdot \|x\|$$

$\forall x \in X$.

- An isometry $f : X \rightarrow Y$ that is an ε -isometry for every $\varepsilon > 0$, i.e. $\|f(x)\| = \|x\| \forall x \in X$.
- A Banach space Y is ε -complemented in X if
 - $Y \subseteq X$
 - $T : X \rightarrow Y$ such that $\|Ty - y\| \leq \varepsilon\|y\| \forall y \in Y$.

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- “0-complemented” means “complemented”.
- f is a $(< \varepsilon)$ -embedding if it is an ε' -isometric embedding for some $0 < \varepsilon' < \varepsilon$.
- Y is $(< \varepsilon)$ -complemented in X if it is ε' -complemented for some $0 < \varepsilon' < \varepsilon$.
- E is *complementably universal* for a class of spaces if every space from the class is isomorphic to a complemented subspace of E .

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Let X be a Banach space.

- A *Schauder decomposition*, (*finite-dimensional decomposition*) is a sequence $P_n : X \rightarrow X$ of finite rank pairwise orthogonal linear operators such that $x = \sum_{n=0}^{\infty} P_n x$ for every $x \in X$. Given such a decomposition, let $Q_n = P_0 + \dots + P_{n-1}$. Then Q_n is a finite-rank projection $Q_n : X \rightarrow X$.

We shall say that X has k -FDD, if $k \geq \sup_{n \in \omega} \|Q_n\|$. We consider 1-FDD only (called *monotone FDD* or *monotone Schauder decomposition*). Every Schauder decomposition is determined by finite-rank projections Q_n such that $Q_n Q_m = Q_{\min(n,m)}$ and $x = \lim_{n \rightarrow \infty} Q_n x$ for $x \in X$.

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Fix $\varepsilon > 0$ and fix a surjective linear operator $f : X \rightarrow Y$ such that

$$(1 + \varepsilon)^{-1} \|x\| \leq \|f(x)\| \leq \|x\|$$

for $x \in X$. Consider the following category $\mathfrak{R}_f^\varepsilon$. The objects:

$i : X \rightarrow Z, j : Y \rightarrow Z$ such that

- $\|i\| \leq 1$ and $\|j\| \leq 1$;
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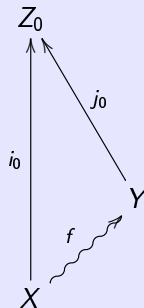
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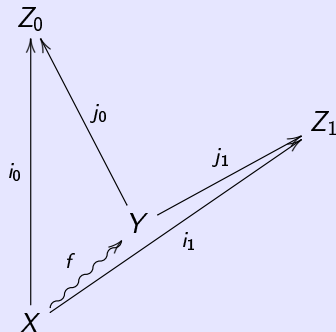
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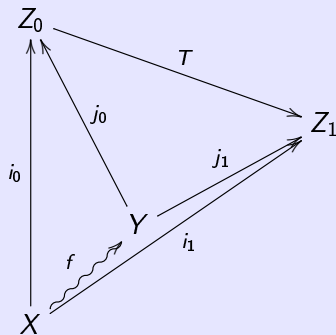
An arrow.



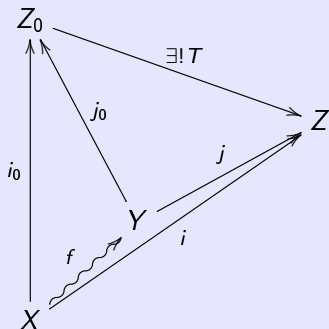
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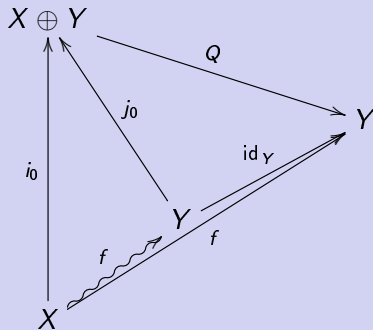
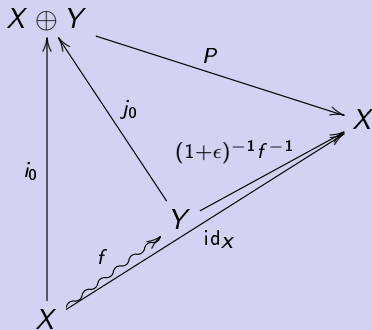


An initial object.



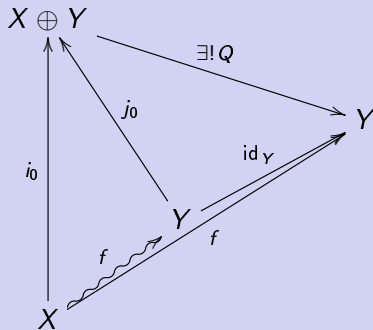
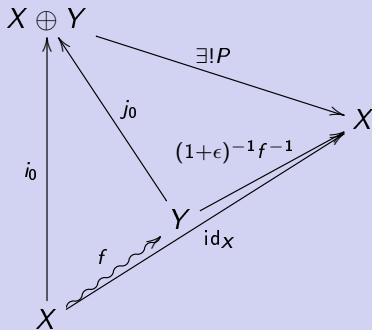
Lemma 1

The category $\mathcal{R}_f^\varepsilon$ has an initial object (i_0, j_0) such that both i_0, j_0 are canonical isometric embeddings into $X \oplus Y$ with a suitable norm $\|\cdot\|$ and there exist projections $P : X \oplus Y \rightarrow X$ and $Q : X \oplus Y \rightarrow Y$ ($\|P\| \leq 1$ and $\|Q\| \leq 1$).



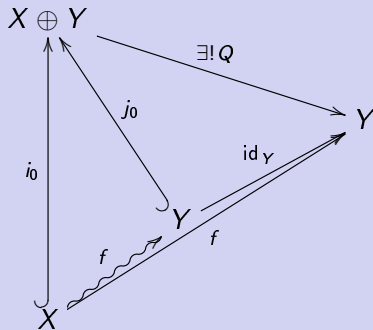
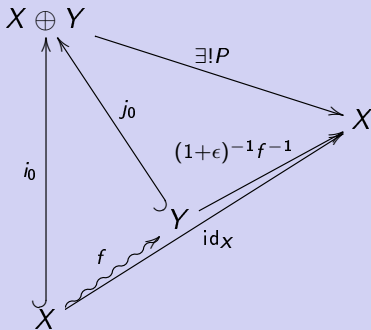
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1) Define

$$G = \{(x, -f(x)) \in X \times Y : x \in \varepsilon^{-1}B_X\}.$$

2) Let K be the convex hull of $(B_X \times \{0\}) \cup (\{0\} \times B_Y) \cup G$.

We will show that the norm

$$\|(x, y)\|_K = \inf\{\|x_0\|_X + \|y_1\|_Y + \varepsilon\|x_2\|_X : (x, y) = (x_0, 0) + (0, y_1) + (x_2, -f(x_2)), (x, y) \in K\},$$

is as required.

Define $i_0(x) = (x, 0)$, $j_0(y) = (0, y)$.

- Firstly we show that (i_0, j_0) is an object of $\mathfrak{K}_f^\varepsilon$:

- $\|i_0\|_K \leq 1$ and $\|j_0\|_K \leq 1$;
- $\|i_0(x) - j_0(f(x))\|_K \leq \varepsilon\|x\|$ for $x \in X$;

- We prove that i_0 and j_0 are isometric embeddings.

Next step is to show that (i_0, j_0) is an initial object of $\mathfrak{K}_f^\varepsilon$.

- Given an object (i, j) of $\mathfrak{K}_f^\varepsilon$, define

$$T(x, y) = i(x) + j(y).$$

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- $P(x, y) = x + (1 + \varepsilon)^{-1}f^{-1}(y)$
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Let \mathfrak{K} be a category. A *Fraïssé sequence* in \mathfrak{K} is an inductive sequence \vec{U} satisfying the following conditions:

(U) For every $A \in \mathfrak{K}$ there exists $n \in \mathbb{N}$ such that $\mathfrak{K}(A, U_n) \neq \emptyset$;

$$U_0 \longrightarrow \dots \longrightarrow U_n \longrightarrow \dots$$

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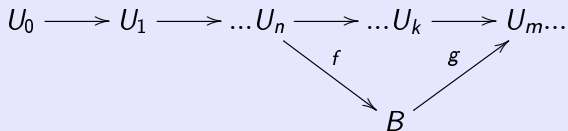
$$U_0 \longrightarrow \dots \longrightarrow U_n \longrightarrow \dots$$

The diagram shows a sequence of objects U_0, \dots, U_n, \dots connected by right-pointing arrows. Below the object U_n , there is an object A . An arrow points from A up and to the right to U_n .

(A) For every $n \in \mathbb{N}$ and for every morphism $f \in \mathfrak{K}(U_n, B)$, where $B \in \mathfrak{K}$, there exist $m \in \mathbb{N}$, $m > n$ and $g \in \mathfrak{K}(B, U_m)$ such that $u_n^m = g \circ f$.

$$\begin{array}{ccccccc}
 U_0 & \longrightarrow & U_1 & \longrightarrow & \dots & U_n & \longrightarrow & \dots & U_k & \longrightarrow & U_m \dots \\
 & & & & & & \searrow & & & & \\
 & & & & & & & f & & & \\
 & & & & & & & & & & B
 \end{array}$$

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We now define the relevant category \mathfrak{K} . The objects of \mathfrak{K} are rational finite-dimensional Banach spaces.

$$X_0 \quad X_1 \quad \dots \quad X_n \quad \dots$$

Given rational finite-dimensional spaces E, F , an \mathfrak{K} -arrow is a pair (e, P) of rational linear operators $e : E \rightarrow F, P : F \rightarrow E$ such that:

(P1) e is a rational isometric embedding.

(P2) $P \circ e = \text{id}_E$ and $\|P\| \leq 1$, where E is the domain of e .

Now we use the fact that every countable category with amalgamations has a Fraïssé sequence.

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$$X_0 \begin{array}{c} \xrightarrow{e_0^1} \\ \xleftarrow{P_0^1} \end{array} X_1 \begin{array}{c} \xrightarrow{e_1^2} \\ \xleftarrow{P_1^2} \end{array} \dots \begin{array}{c} \xrightarrow{e_{n-1}^n} \\ \xleftarrow{P_{n-1}^n} \end{array} X_n \begin{array}{c} \xrightarrow{e_n^{n+1}} \\ \xleftarrow{P_n^{n+1}} \end{array} \dots$$

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Let us consider the following extension property of a Banach space X :

- (E) Given a pair $E \subseteq F$ of finite-dimensional Banach spaces such that E is complemented in F , given an isometric embedding $i : E \rightarrow X$ such that $i[E]$ is complemented in X , for every $\varepsilon > 0$ there exists an ε -isometric embedding $g : F \rightarrow X$ such that $\|g \upharpoonright E - i\| < \varepsilon$ and $g[F]$ is ε -complemented in X .

Theorem (Uniqueness)

Let \mathbb{P} and \mathbb{K} be Banach spaces satisfying condition (E) and let $h : A \rightarrow B$ be a bijective linear isometry between complemented finite-dimensional subspaces of \mathbb{P} and \mathbb{K} , respectively. Then for every $\varepsilon > 0$ there exists a bijective linear isometry $H : \mathbb{P} \rightarrow \mathbb{K}$ that is ε -close to h . In particular, \mathbb{P} and \mathbb{K} are linearly isometric.

$$\begin{array}{ccc}
 \mathbb{P} & & \mathbb{K} \\
 \updownarrow & & \updownarrow \\
 A & \xrightarrow{h} & B
 \end{array}$$

Theorem (Uniqueness)

Let \mathbb{P} and \mathbb{K} be Banach spaces satisfying condition (E) and let $h : A \rightarrow B$ be a bijective linear isometry between complemented finite-dimensional subspaces of \mathbb{P} and \mathbb{K} , respectively. Then for every $\varepsilon > 0$ there exists a bijective linear isometry $H : \mathbb{P} \rightarrow \mathbb{K}$ that is ε -close to h . In particular, \mathbb{P} and \mathbb{K} are linearly isometric.

$$\begin{array}{ccc}
 \mathbb{P} & \xrightarrow{H} & \mathbb{K} \\
 \updownarrow & & \updownarrow \\
 A & \xrightarrow{h} & B
 \end{array}$$

Theorem (Universality)

Let X be a Banach space with a monotone FDD. Then there exists an isometric embedding $e : X \rightarrow \mathbb{P}$ such that $e[X]$ is 1-complemented in \mathbb{P} .



J. Garbulińska, *Isometric uniqueness of a complementably universal Banach space for Schauder decompositions*,