Isometric uniqueness of a complementably universal Banach space for Schauder decomposition.

Joanna Garbulińska

Jan Kochanowski University in Kielce/Jagiellonian University in Kraków, Poland

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We present an isometric version of the complementably universal Banach space $\mathbb{P}$ with a Schauder decomposition. The space $\mathbb{P}$ is isomorphic to Pełczyński's space with a universal basis as well as to Kadec's complementably universal space with the bounded approximation property.

In 1969 Pełczyński constructed a complementably universal Banach space with a Schauder basis. Two years later, Kadec constructed a complementably universal Banach space for the class of spaces with the BAP. Just after, Pełczyński showed that every Banach space with BAP is complemented in a space with a basis. Applying Pełczyński' decomposition argument, one immediately concludes that both spaces are isomorphic.

- Let X, Y be Banach spaces, $\varepsilon > 0$. $f : X \to Y$ is an *ε-isometry* if

$$(1+\varepsilon)^{-1} \cdot \|x\| \leq \|f(x)\| \leq (1+\varepsilon) \cdot \|x\|$$

  $\forall x \in X$.

- An *isometry* $f : X \to Y$ that is an ε-isometry for every $\varepsilon > 0$, i.e. $\|f(x)\| = \|x\| \; \forall x \in X$.

- A Banach space Y is *ε-complemented* in X if
  - $Y \subseteq X$
  - $T : X \to Y$ such that $\|Ty - y\| \leq \varepsilon \|y\| \; \forall y \in Y$.

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- "0-complemented" means "complemented".
- f is a $(<\varepsilon)$-*embedding* if it is an ε'-isometric embedding for some $0 < \varepsilon' < \varepsilon$.
- Y is $(<\varepsilon)$-*complemented* in X if it is ε'-complemented for some $0 < \varepsilon' < \varepsilon$.
- E is *complementably universal* for a class of spaces if every space from the class is isomorphic to a complemented subspace of E.

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Let X be a Banach space.

- A *Schauder decomposition*, (*finite-dimensional decomposition*) is a sequence $P_n : X \to X$ of finite rank pairwise orthogonal linear operators such that $x = \sum_{n=0}^{\infty} P_n x$ for every $x \in X$. Given such a decomposition, let $Q_n = P_0 + \cdots + P_{n-1}$. Then Q_n is a finite-rank projection $Q_n : X \to X$.

We shall say that X has *k-FDD*, if $k \geq \sup_{n\in\omega} \|Q_n\|$. We consider 1-FDD only (called *monotone FDD* or *monotone Schauder decomposition*). Every Schauder decomposition is determined by finite-rank projections Q_n such that $Q_n Q_m = Q_{\min(n,m)}$ and $x = \lim_{n\to\infty} Q_n x$ for $x \in X$.

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Fix $\varepsilon > 0$ and fix a surjective linear operator $f : X \to Y$ such that

$$(1+\varepsilon)^{-1}\|x\| \le \|f(x)\| \le \|x\|$$

for $x \in X$. Consider the following category $\mathfrak{K}_f^\varepsilon$. The objects: $i : X \to Z$, $j : Y \to Z$ such that

- $\|i\| \le 1$ and $\|j\| \le 1$;
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An arrow.

$$\begin{tikzcd}
Z_0 & \\
 & Y \arrow[ul, "j_0"'] \\
X \arrow[uu, "i_0"] \arrow[ur, rightsquigarrow, "f"] &
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An initial object.

$$\begin{tikzcd}
Z_0 \arrow[drr, "\exists ! T"] & & \\
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Lemma1

The category $\mathfrak{K}_f^\varepsilon$ has an initial object (i_0, j_0) such that both i_0, j_0 are canonical isometric embeddings into $X \oplus Y$ with a suitable norm $\|\cdot\|$ and there exist projections $P : X \oplus Y \to X$ and $Q : X \oplus Y \to Y$ ($\|P\| \leq 1$ and $\|Q\| \leq 1$).

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\begin{tikzcd}
X \oplus Y \arrow[rrr, "P"] & & & X \\
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1) Define

$$G = \{(x, -f(x)) \in X \times Y : x \in \varepsilon^{-1} B_X\}.$$

2) Let K be the convex hull of $(B_X \times \{0\}) \cup (\{0\} \times B_Y) \cup G$.
We will show that the norm
$\|(x,y)\|_K = \inf\{\|x_0\|_X + \|y_1\|_Y + \varepsilon\|x_2\|_X : (x,y) = (x_0, 0) + (0, y_1) + (x_2, -f(x_2)), (x,y) \in K\}$, is as required.
Define $i_0(x) = (x, 0)$, $j_0(y) = (0, y)$.

- Firstly we show that (i_0, j_0) is an object of $\mathfrak{K}_f^\varepsilon$:
  - $\|i_0\|_K \leq 1$ and $\|j_0\|_K \leq 1$;
  - $\|i_0(x) - j_0(f(x))\|_K \leq \varepsilon\|x\|$ for $x \in X$;
- We prove that i_0 and j_0 are isometric embeddings.

Next step is to show that (i_0, j_0) is an initial object of $\mathfrak{K}_f^\varepsilon$.

- Given an object (i, j) of $\mathfrak{K}_f^\varepsilon$, define

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3) Define linear operators $P : X \oplus Y \to X$ and $Q : X \oplus Y \to Y$ as:

- $P(x, y) = x + (1 + \varepsilon)^{-1} f^{-1}(y)$
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Let $\mathfrak{K}$ be a category. A *Fraïssé sequence* in $\mathfrak{K}$ is an inductive sequence $\vec{U}$ satisfying the following conditions:
(U) For every $A \in \mathfrak{K}$ there exists $n \in \mathbb{N}$ such that $\mathfrak{K}(A, U_n) \neq \emptyset$;

$$U_0 \longrightarrow \dots \longrightarrow U_n \longrightarrow \dots$$

$$A$$

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$$\begin{tikzcd} U_0 \arrow[r] & \dots \arrow[r] & U_n \arrow[r] & \dots \\ & A \arrow[ur] & & \end{tikzcd}$$

(A) For every $n \in \mathbb{N}$ and for every morphism $f \in \mathfrak{K}(U_n, B)$, where $B \in \mathfrak{K}$, there exist $m \in \mathbb{N}$, $m > n$ and $g \in \mathfrak{K}(B, U_m)$ such that $u_n^m = g \circ f$.

$$\begin{tikzcd}
U_0 \arrow[r] & U_1 \arrow[r] & \ldots U_n \arrow[r] \arrow[dr, "f"] & \ldots U_k \arrow[r] & U_m \ldots \\
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We now define the relevant category $\mathfrak{K}$. The objects of $\mathfrak{K}$ are rational finite-dimensional Banach spaces.

$$X_0 \qquad X_1 \qquad \dots \qquad X_n \qquad \dots$$

Given rational finite-dimensional spaces E, F, an $\mathfrak{K}$-arrow is a pair (e, P) of rational linear operators $e : E \to F$, $P : F \to E$ such that:

(P1) e is a rational isometric embedding.

(P2) $P \circ e = \mathrm{id}_E$ and $\|P\| \leq 1$, where E is the domain of e.

Now we use the fact that every countable category with amalgamations has a Fraïssé sequence.

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$$X_0 \underset{P_0^1}{\overset{e_0^1}{\rightleftarrows}} X_1 \underset{P_1^2}{\overset{e_1^2}{\rightleftarrows}} \ldots \underset{P_{n-1}^n}{\overset{e_{n-1}^n}{\rightleftarrows}} X_n \underset{P_n^{n+1}}{\overset{e_n^{n+1}}{\rightleftarrows}} \ldots$$

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Let us consider the following extension property of a Banach space X:

(E) Given a pair $E \subseteq F$ of finite-dimensional Banach spaces such that E is complemented in F, given an isometric embedding $i : E \to X$ such that $i[E]$ is complemented in X, for every $\varepsilon > 0$ there exists an ε-isometric embedding $g : F \to X$ such that $\|g \restriction E - i\| < \varepsilon$ and $g[F]$ is ε-complemented in X.

Theorem (Uniqueness)

Let $\mathbb{P}$ and $\mathbb{K}$ be Banach spaces satisfying condition (E) and let $h : A \to B$ be a bijective linear isometry between complemented finite-dimensional subspaces of $\mathbb{P}$ and $\mathbb{K}$, respectively. Then for every $\varepsilon > 0$ there exists a bijective linear isometry $H : \mathbb{P} \to \mathbb{K}$ that is ε-close to h. In particular, $\mathbb{P}$ and $\mathbb{K}$ are linearly isometric.

$$\begin{tikzcd}
\mathbb{P} \arrow[d, shift left, two heads] & \mathbb{K} \arrow[d, shift left, two heads] \\
A \arrow[u, shift left] \arrow[r, "h"] & B \arrow[u, shift left]
\end{tikzcd}$$

Theorem (Uniqueness)

Let $\mathbb{P}$ and $\mathbb{K}$ be Banach spaces satisfying condition (E) and let $h : A \to B$ be a bijective linear isometry between complemented finite-dimensional subspaces of $\mathbb{P}$ and $\mathbb{K}$, respectively. Then for every $\varepsilon > 0$ there exists a bijective linear isometry $H : \mathbb{P} \to \mathbb{K}$ that is ε-close to h. In particular, $\mathbb{P}$ and $\mathbb{K}$ are linearly isometric.

$$
\begin{tikzcd}
\mathbb{P} \arrow[r, "H"] \arrow[d, shift left, two heads] & \mathbb{K} \arrow[d, shift left, two heads] \\
A \arrow[r, "h"] \arrow[u, shift left] & B \arrow[u, shift left]
\end{tikzcd}
$$

Theorem (Universality)

Let X be a Banach space with a monotone FDD. Then there exists an isometric embedding $e : X \to \mathbb{P}$ such that $e[X]$ is 1-complemented in $\mathbb{P}$.

J. Garbulińska, *Isometric uniqueness of a complementably universal Banach space for Schauder decompositions*,