Topological classification of countable IFS-attractors

Magdalena Nowak

The Jan Kochanowski University in Kielce / Jagiellonian University in Krakow

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IFS

Definition

If (X, d) is a complete metric space and $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$ is a collection of (weak) contractions of X to itself, then $\mathcal{F}$ is said to be an **Iterated Function System** (IFS).

A map $f : X \to X$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that for any $x, y \in X$

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y).$$

A map $f : X \to X$ is called a *weak contraction* if for each $x \neq y$, $x, y \in X$

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IFS-attractors

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The **attractor of the IFS** $\mathcal{F}$ (*IFS-attractor*) it is the unique nonempty compact set $\mathcal{K} \subset X$ which is invariant by the IFS $\mathcal{F}$, in the sense:

$$\mathcal{K} = f_1(\mathcal{K}) \cup f_2(\mathcal{K}) \cup \cdots \cup f_n(\mathcal{K}).$$

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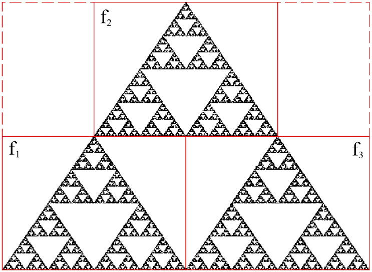
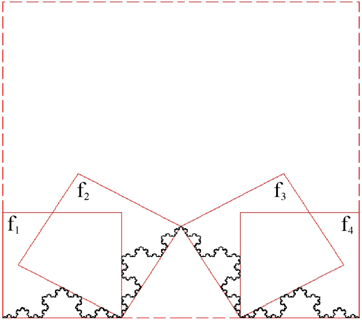
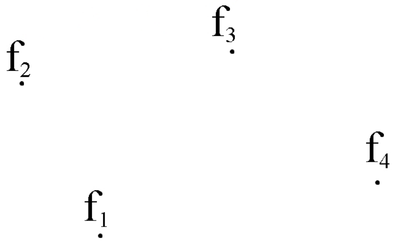




Countable sets

Remark

Every finite set is an IFS-attractor in every metric.



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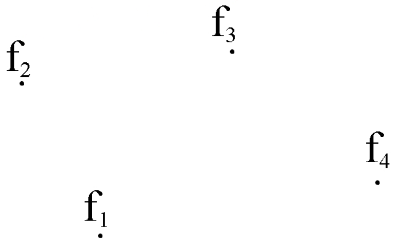

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Scattered space

A compact metric space is countable iff it is scattered.

Definition

A space X is called **scattered** iff every non-empty subspace Y has an isolated point in Y.

Mazurkiewicz-Sierpiński theorem

Every countable compact scattered space X is homeomorphic to the space $\omega^{\beta} \cdot n + 1$ with the order topology, where $\beta = \mathrm{ht}(X)$ and $n = |X^{(\beta)}|$ is finite.

$$X \backsim \omega^{\beta} \cdot n + 1 \backsim (\omega^{\beta} + 1) \cdot n$$

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Cantor-Bendixson level

For a scattered space X let

$$X' = \{x \in X | x \text{ is an accumulation point of } X\}$$

be the Cantor-Bendixson derivative of X.

- $X^{(\alpha+1)} = (X^{(\alpha)})'$
- $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ for a limit ordinal α.

Definition

The height of X is $\mathrm{ht}(X) = \min\{\beta | X^{(\beta)} \text{ is finite}\}$.

Definition

For each element x of scattered space X, we define its Cantor-Bendixson rank as

$$\mathrm{rk}(x) = \alpha \text{ such that } x \in X^{(\alpha)} \setminus X^{(\alpha+1)}.$$

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Some properties of height and Cantor-Bendixson rank.

For U and V scattered compact space:

- if $U \subset V$ then $\mathrm{ht}(U) \leq \mathrm{ht}(V)$
- $\mathrm{ht}(U \cup V) = \max(\mathrm{ht}(U), \mathrm{ht}(V))$
- $\mathrm{ht}(f(U)) \leq \mathrm{ht}(U)$ for every continuous function f
- $\mathrm{ht}(U) \geq \mathrm{rk}(x)$ for every open neighborhood U of x

Topological classification of countable IFS-attractors

Theorem (M.N.)

If $X = \omega^{\beta} \cdot n + 1$ be a scattered, compact, metric space and $n \geq 1$, then

1. $\beta = 0 \Rightarrow X$ is IFS-attractor in every metric
2. $\beta = \alpha + 1 \Rightarrow$ there exist $f, g \colon X \to \mathbb{R}$ homeomorphisms such that
   1. $f(X)$ is an IFS-attractor
   2. $g(X)$ is not an attractor of any weak IFS
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$f(X)$ is contractive IFS-attractor when $\beta = \alpha + 1$

Theorem

For every $\varepsilon > 0$ and every countable ordinal α the scattered space $\omega^{\alpha+1} + 1$ is homeomorphic to the attractor of an iterated function system consisting of two contractions $\{\varphi, \varphi_{\alpha+1}\}$ in the unit interval $I = [0, 1]$, such that

$$\max(\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi_{\alpha+1})) < \varepsilon.$$

Lemma

If A and B are disjoint IFS-attractors then $A \cup B$ is also IFS-attractor.

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Successor height: $\omega^{\alpha+1}+1$

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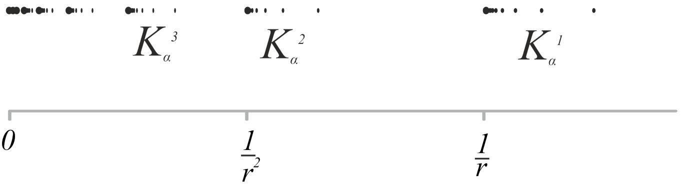
$K_{\alpha+1} = \{0\} \cup \bigcup_{i\in\mathbb{N}} K_\alpha^i$ and $r > 3$.



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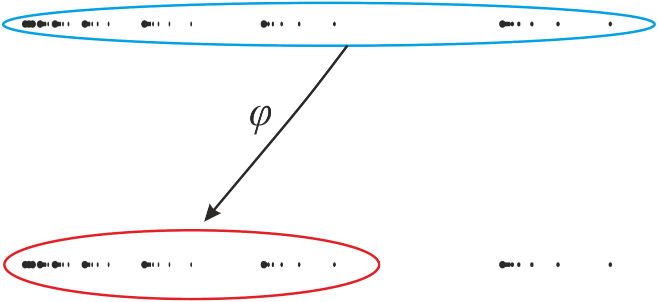

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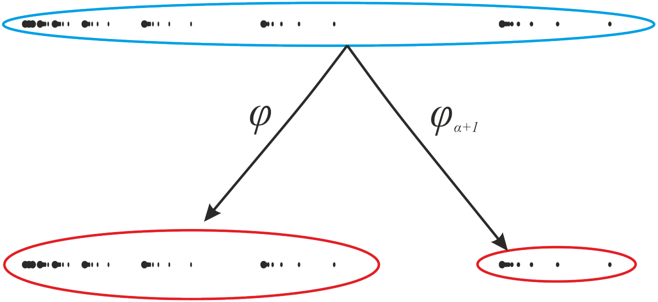

$\varphi(x) = \frac{x}{r}$ $\qquad \varphi_{\alpha+1}(K_{\alpha+1}) = K_\alpha^1$ and $\mathrm{Lip}(\varphi_{\alpha+1}) = \frac{1}{r-2}$

$g(X)$ is not an attractor of any weak IFS when $\beta = \alpha + 1$

Theorem

There exists a convergent sequence $\mathcal{K} \backsim \omega + 1$ which is not a weak IFS-attractor in $[0,1]$ with standard Euclidean metric.

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There exists a compact scattered metric space $\mathcal{K} \backsim \omega^{\alpha+1} + 1$ which is not a weak IFS-attractor in $[0,1]$ with standard Euclidean metric.

Lemma

If $X = X_0 \cup \ldots \cup X_n$ is a weak IFS-attractor and each X_i is compact and isometric to X_0 and $\operatorname{dist}(X_i, X_j) > \operatorname{diam}(X_0)$ for every $i < j \leq n$ then every X_i is a weak IFS-attractor.

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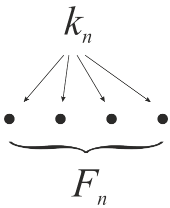
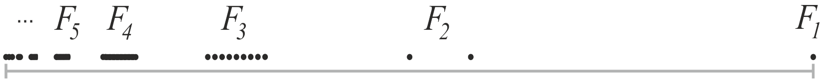
Construction of $\mathcal{K} \backsim \omega + 1$

For $k_1 = 1$ and
$k_n = n(k_{n-1} + \cdots + k_1)$

$\mathrm{diam}(F_n) < \mathrm{dist}(F_n, F_{n+1})$



$$\mathcal{K} := \{0\} \cup \bigcup_{n=1}^{\infty} F_n \ .$$



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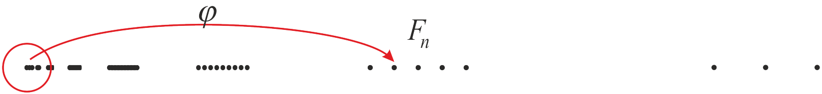

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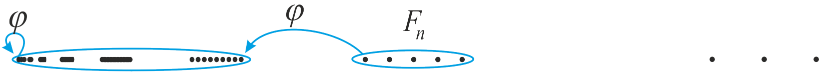

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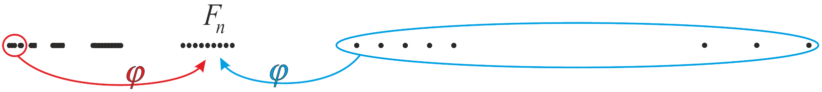

End of the proof

Suppose that $\mathcal{K}$ is an IFS-attractor. $\mathcal{K} = \bigcup_{i=1}^{r} \varphi_i(\mathcal{K})$

We can write the set $\mathcal{K}$ as the union
$\mathcal{K} = \bigcup_{i=1}^{m} \varphi_i(\mathcal{K}) \cup \bigcup_{i=m+1}^{r} \varphi_i(\mathcal{K})$

For almost every n we have $F_n \subset \bigcup_{i=1}^{m} \varphi_i(F_{n-1} \cup \cdots \cup F_1)$ so

$$k_n = |F_n| \leq |\bigcup_{i=1}^{m} \varphi_i(F_{n-1} \cup \cdots \cup F_1)| \leq m(k_{n-1} + \cdots + k_1) \ .$$

But $k_n = n(k_{n-1} + \cdots + k_1)$ so for $n > m$ we get a contradiction.

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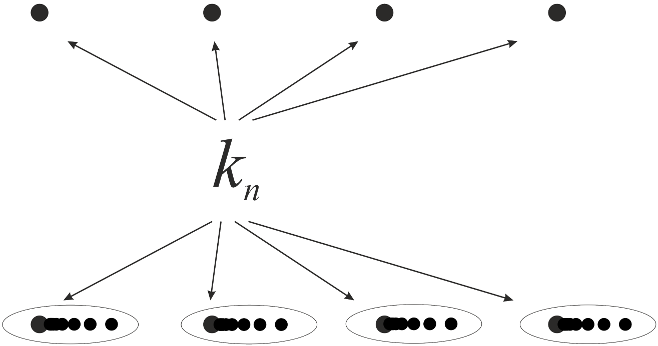
Generalization of F_n



$$\mathrm{diam}(F_n) < \mathrm{dist}(F_n, F_{n+1})$$

X is never IFS-attractor when $\mathrm{ht}(X) = \beta > 0$ limit ordinal

Theorem

For a limit ordinal β the scattered space $\mathcal{K} \backsim \omega^\beta \cdot n + 1$ is not homeomorphic to any IFS-attractor consisting of weak contractions.

Proof:
Suppose that $\mathcal{K} \backsim \omega^\beta \cdot n + 1$ has a fixed metric d and there exists IFS $\mathcal{F}$ such that $\mathcal{K} = \bigcup_{f \in \mathcal{F}} f(\mathcal{K})$.
Denote by $D = \mathcal{K}^{(\beta)}$ a set of points from $\mathcal{K}$ of Cantor-Bendixson rank β. $n = |D|$.

1. $\mathcal{F}_1 := \{f \in \mathcal{F} | \, \mathrm{ht}(f(\mathcal{K})) = \beta\}$
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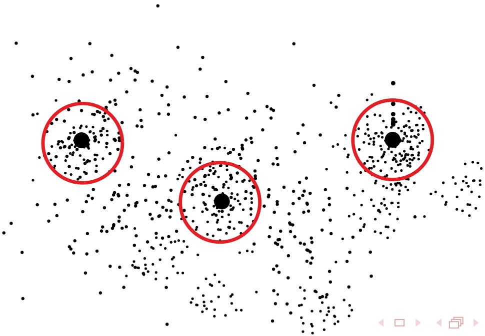
Proof

Consecutive derivatives $\mathcal{K}^{(\rho)}$ cluster around $D = \bigcap_{\rho<\beta} \mathcal{K}^{(\rho)}$ so there exists ρ such that $\max_{f \in \mathcal{F}_0} \mathrm{ht}(f(\mathcal{K})) < \rho < \beta$ and for $|D| > 1$

$$\mathcal{K}^{(\rho)} \subset \bigcup_{x \in D} B(x, \frac{\varepsilon}{2})$$

where $\varepsilon = \min\{d(x,y) |\ x \neq y; x, y \in D \cup \bigcup_{f \in \mathcal{F}_1} f(D)\} > 0.$



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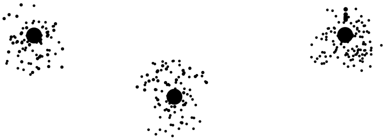

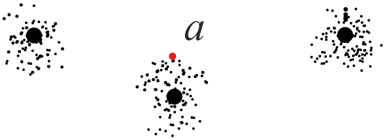

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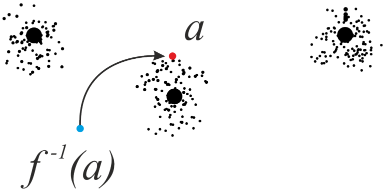

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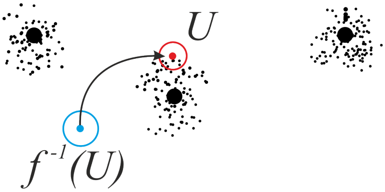

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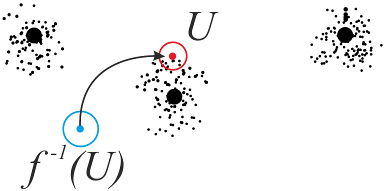

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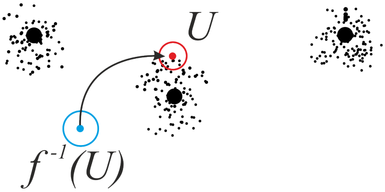

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