A new space $C(K)$ with few operators

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The problem

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**Question (Haïly, Kaidi, Rodríguez-Palacios)**
Is there an infinite dimensional Banach space $X$ such that every injective operator $T : X \rightarrow X$ is surjective?
Spaces with few operators, like Gowers-Maurey, satisfy:

1. Every surjective operator $T: X \to X$ is injective.
2. Every injective operator $T: X \to X$ with closed range is surjective.

However, if $\{e_n\}$ is a basis of subspace of $X$, $\|e_n\| = 2^{-n}$.

$\{f^*_n\} \subset X^*$ is weak* dense of norm-one operators.

Then $T(x) = \sum_n f^*_n(x) e_n$ defines an injective operator which is not surjective.

Theorem

There exists a compact space $K$ such that every injective operator $T: C(K) \to C(K)$ is surjective.
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There exists a compact space $K$ such that every injective operator $T : C(K) \to C(K)$ is surjective.
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Therefore, our space $K$ must be an almost $P$-space: every nonempty zero set has nonempty interior.

If $K = \text{Stone}(B)$, this means that every decreasing sequence $a_1 > a_2 > \cdots$ in $B$ fails to have an infimum.
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Looking for a compact space $K$

In every space $C(K)$, for every continuous $h : K \rightarrow K$ we have the composition operator $S_h(f) = f \circ h$. 

$S_h$ is injective iff $h$ is surjective.

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So, we need that every surjective $h : K \rightarrow K$ is bijective.

However, killing all non-constant $h : K \rightarrow K$ is not enough to control all operators $C(K) \rightarrow C(K)$. For this, we need $K$ to be a Koszmider space.
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So, we need that every surjective $h : K \to K$ is bijective. However, killing all non-constant $h : K \to K$ is not enough to control all operators $C(K) \to C(K)$. For this, we need $K$ to be a Koszmider space.
A Koszmider space is a space $K$ such that every operator $C(K) \rightarrow C(K)$ is a weak multiplier.
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**Weak multiplications (stronger notion)**

An operator $T : C(K) \rightarrow C(K)$ is a weak multiplication if $T = Tg + S$ where $g \in C(K)$, $S$ is weakly compact.
Schlackow’s approach to Koszmider spaces

Let $B$ be a Boolean algebra such that

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Koszmider spaces

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Let $B$ be a Boolean algebra such that

- for every pairwise disjoint family $\{a_n\} \cup \{b_n\}$, there exists infinite $\tau \subset \omega$ such that

Then $\text{Stone}(B)$ is a Koszmider space.

The almost $P$-space condition is incompatible with countable suprema to exist in $B$. 
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**Schlackow’s approach to Koszmider spaces**
More general approach

Let $B$ be a Boolean algebra such that for every $\{a_n\}$ there exists a morphism $\phi: A \subset P(\omega) \to B$ such that $\phi(\{n\}) = a_n$, for every $\{b_n\}$, there exists infinite $\tau \subset \omega$ such that $\{b_n: n \in \tau\}$ and $\{b_n: n \notin \tau\}$ are not separated. Then $Stone(B)$ is a Koszmider space.
More general approach

Let $B$ be a Boolean algebra such that for every $\{a_n\}$ there exists a morphism $\phi: \mathcal{P}(\omega) \rightarrow B$ such that $\phi(\{n\}) = a_n$. For every $\{b_n\}$, there exists infinite $\tau \subset \omega$ such that $\tau \in \mathcal{A}$, $\{b_n : n \in \tau\}$ and $\{b_n : n \notin \tau\}$ are not separated. Then $\text{Stone}(B)$ is a Koszmider space.
Koszmider spaces

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Then $\text{Stone}(B)$ is a Koszmider space.
Our space

Theorem

There exists $K = \text{Stone}(B)$ that is a Koszmider space and an almost $P$-space. Every injective $T : C(K) \to C(K)$ is surjective.