A new space $C(K)$ with few operators

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The problem

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Question (Haïly, Kaidi, Rodríguez-Palacios)

Is there an infinite dimensional Banach space X such that every injective operator $T : X \longrightarrow X$ is surjective?

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Theorem

There exists a compact space K such that every injective operator $T : C(K) \longrightarrow C(K)$ is surjective.

Looking for a compact space

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If $K = Stone(B)$, this means that every decreasing sequence $a_1 > a_2 > \cdots$ in B fails to have an infimum.

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So, we need that every surjective $h : K \longrightarrow K$ is bijective. However, killing all non-constant $h : K \longrightarrow K$ is not enough to control all operators $C(K) \longrightarrow C(K)$. For this, we need K to be a Koszmider space.

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Weak multiplications (stronger notion)

An operator $T : C(K) \longrightarrow C(K)$ is a weak multiplication if $T = Tg + S$ where $g \in C(K)$, S is weakly compact.

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The almost P-space condition is incompatible with countable suprema to exist in B.

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Our space

Theorem

There exists $K = Stone(B)$ that is a Koszmider space and an almost P-space. Every injective $T : C(K) \longrightarrow C(K)$ is surjective.