Freie Universität Berlin

Some new results on union ultrafilters

P. Krautzberger
Freie Universität Berlin

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Outline

What are union ultrafilters?

What can union ultrafilters do for you?

What can you do for union ultrafilters?

Hindman's Theorem

If $\mathbb{N}$ is finitely coloured, there exist infinite $\mathbf{x} = (x_i)_{i<\omega}$ such that

$$FS(\mathbf{x}) = \{x_{i_0} + \ldots + x_{i_k} \mid k < \omega\}$$

is monochromatic.

Hindman's Theorem

If $[\omega]^{<\omega}$ is finitely coloured, there exist infinite disjoint $\mathbf{s} = (s_i)_{i<\omega}$ such that

$$FU(\mathbf{s}) = \{s_{i_0} \cup \ldots \cup s_{i_k} \mid k < \omega\}$$

is monochromatic.

Summable Ultrafilter

$p \in \beta\mathbb{N}$ is a *summable ultrafilter* if it has a base of $FS(\mathbf{x})$-sets.

Union Ultrafilter

$u \in \beta[\omega]^{<\omega}$ is a *union ultrafilter* if it has a base of $FU(\mathbf{s})$-sets.

Definition (Variants of Union Ultrafilters)

A union ultrafilter u on $[\omega]^{<\omega}$ is called

- *ordered* if it has a base of $FU(\mathbf{s})$-sets s.t. $s_i \ll s_j$ $(i < j)$.
- *stable* if for every sequence $FU(\mathbf{s}^n)$ $(n < \omega)$ in u there exists $FU(\mathbf{t}) \in u$ such that

$$\mathbf{t} \subseteq^* FU(\mathbf{s}^n) \quad (n < \omega).$$

Existence

Some classical results (Blass, Hindman)

- If u is a union ultrafilter, then $\min(u)$, $\max(u)$ are (rapid) P-points.
- If u is an ordered union ultrafilter, then $\min(u)$, $\max(u)$ are Ramsey.
- Assuming e.g. $cov(\mathscr{M}) = \mathfrak{c}$, union ultrafilters exist.

More applications

Algebra

- Almost all summables have ”trivial sums“ (P.K.)
- Summables are strongly right maximal, i.e., $\{q \in \beta\mathbb{N} \mid q + p = p\} = \{p\}$ (folklore)

Topology

- Orbits of summables are vD-spaces (Protasov)
- Union ultrafilters generate maximal group topologies on $[\omega]^{<\omega}$ (Protasov)

Set theory

- $Con(\mathfrak{u} < \mathfrak{g})$ via iterated Matet forcing (Blass)
- $NCF \not\Rightarrow FD$ (Mildenberger, Shelah)

Unordered union ultrafilters (P.K.)

Nearly ordered union ultrafilters

Assuming *cov*$(\mathscr{M}) = \mathfrak{c}$, there exist stable unordered union ultrafilters with min and max Ramsey.

Stability as Ramsey Property (Blass, P.K.)

A union ultrafilter is stable iff

Whenever $([\omega]^{<\omega})^2$ is finitely coloured there exists $FU(\mathbf{s}) \in u$ such that

$$\{(v, w) \in FU(\mathbf{s})^2 \mid v < w\}$$

is monochromatic.

Question

Can there be union ultrafilters that are not stable?