Some new results on union ultrafilters

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Outline

What are union ultrafilters?

What can union ultrafilters do for you?

What can you do for union ultrafilters?
Hindman’s Theorem

If $\mathbb{N}$ is finitely coloured, there exist infinite $\mathbf{x} = (x_i)_{i<\omega}$ such that

$$FS(\mathbf{x}) = \{x_{i_0} + \ldots + x_{i_k} \mid k < \omega\}$$

is monochromatic.

Summable Ultrafilter

$p \in \beta\mathbb{N}$ is a summable ultrafilter if it has a base of $FS(\mathbf{x})$-sets.

Hindman’s Theorem

If $[\omega]^{<\omega}$ is finitely coloured, there exist infinite disjoint $\mathbf{s} = (s_i)_{i<\omega}$ such that

$$FU(\mathbf{s}) = \{s_{i_0} \cup \ldots \cup s_{i_k} \mid k < \omega\}$$

is monochromatic.

Union Ultrafilter

$u \in \beta[\omega]^{<\omega}$ is a union ultrafilter if it has a base of $FU(\mathbf{s})$-sets.
Definition (Variants of Union Ultrafilters)

A union ultrafilter $u$ on $[\omega]<\omega$ is called

- **ordered** if it has a base of $FU(s)$-sets s.t. $s_i \ll s_j$ ($i < j$).
- **stable** if for every sequence $FU(s^n)$ ($n < \omega$) in $u$ there exists $FU(t) \in u$ such that

$$t \subseteq^* FU(s^n) \quad (n < \omega).$$
Existence

Some classical results (Blass, Hindman)

- If \( u \) is a union ultrafilter, then \( \min(u) \), \( \max(u) \) are (rapid) \( P \)-points.
- If \( u \) is an ordered union ultrafilter, then \( \min(u) \), \( \max(u) \) are Ramsey.
- Assuming e.g. \( \text{cov}(\mathcal{M}) = c \), union ultrafilters exist.
More applications

**Algebra**
- Almost all summables have "trivial sums" (P.K.)
- Summables are strongly right maximal, i.e.,
  \[ \{ q \in \beta \mathbb{N} \mid q + p = p \} = \{ p \} \] (folklore)

**Topology**
- Orbits of summables are vD-spaces (Protasov)
- Union ultrafilters generate maximal group topologies on \([\omega]^{<\omega}\) (Protasov)

**Set theory**
- \( \text{Con}(\mu < \mathfrak{g}) \) via iterated Matet forcing (Blass)
- \( NCF \nRightarrow FD \) (Mildenberger, Shelah)
Nearly ordered union ultrafilters

Assuming \( \text{cov}(\mathcal{M}) = \mathfrak{c} \), there exist stable unordered union ultrafilters with min and max Ramsey.
A union ultrafilter is stable iff

Whenever \([\omega]^{<\omega})^2\) is finitely coloured there exists \(FU(s) \in u\) such that

\[\{(v, w) \in FU(s)^2 \mid v < w\}\]

is monochromatic.

Question

Can there be union ultrafilters that are not stable?