Representing ideals on Polish spaces

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Joint work with Marcin Sabok
Definition

Suppose that $X$ is a Polish space and $I$ is a $\sigma$-ideal on $X$ containing all singletons. Given a dense countable set $D \subset X$ we define the ideal

$$J_I = \{ a \subset D : cl(a) \in I \}.$$
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Suppose that $X$ is a Polish space and $I$ is a $\sigma$-ideal on $X$ containing all singletons. Given a dense countable set $D \subset X$ we define the ideal

$$J_I = \{ a \subset D : \text{cl}(a) \in I \}.$$ 

Given an ideal $J$ on a countable set $E$ we say that $J$ is represented on a Polish space if there are $X$, $I$, $D$ as above and a bijection $\rho : E \to D$ such that $J = \{ a \subset E : \rho[a] \in J_I \}$. 

Examples

$NWD(Q) = \{ a \subset Q \cap [0,1] : a \text{ is nowhere dense} \}$

$\text{NULL}(Q) = \{ a \subset Q \cap [0,1] : \text{cl}(a) \text{ is of Lebesgue measure zero} \}$

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Ideals represented on Polish spaces

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For an ideal $J$ on $\omega$ the equivalence relation $E_J$ on $2^\omega$ is given by

$$x E_J y \iff x \triangle y \in J.$$
Motivation

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For an ideal $J$ on $\omega$ the equivalence relation $E_J$ on $2^\omega$ is given by $x E_J y \iff x \triangle y \in J$.

Theorem (Zapletal, 2012)

Let $J$ be an ideal represented on a compact space.

(a) Suppose that $E$ is an equivalence relation of a turbulent action. Every Borel homomorphism from $E$ to $E_J$ maps a comeager set to a single $E_J$-equivalence class.

(b) Suppose that $J$ is represented by a $\Pi^0_2$-ideal of compact sets. Every Borel homomorphism from $E_J$ to countable structures maps a comeager set to a single equivalence class.
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For an ideal \( J \) on \( \omega \) the equivalence relation \( E_J \) on \( 2^\omega \) is given by
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(a) Suppose that \( E \) is an equivalence relation of a turbulent action. Every Borel homomorphism from \( E \) to \( E_J \) maps a comeager set to a single \( E_J \)-equivalence class.
(b) Suppose that \( J \) is represented by a \( \Pi^0_2 \sigma \)-ideal of compact sets. Every Borel homomorphism from \( E_J \) to countable structures maps a comeager set to a single equivalence class.
Conjecture (Sabok-Zapletal)

For any ideal $J$ on a countable set the following are equivalent:
(a) $J$ is represented on a compact space;
(b) $J$ is dense $\Pi^0_3$ and weakly selective.
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Definition

We say that an ideal $J$ on a countable set $D$ is weakly selective if for any $a \notin J$ and any $f : a \to \omega$ there is $b \subset a$ with $b \notin J$ such that $f$ restricted to $b$ is either one-to-one or constant.
Characterization of ideals represented on Polish spaces

Definition

An ideal $J$ on a countable set is dense if any infinite set contains an infinite subset belonging to the ideal.
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We say that an ideal $J$ on a countable set $D$ is countably separated if there is a countable family $\{x_n : n \in \omega\}$ of subsets of $D$ such that for any $a, b \subset D$ with $a \notin J$ and $b \in J$ there is $n \in \omega$ with $a \cap x_n \notin J$ and $b \cap x_n = \emptyset$. 
Main Theorem (K.-Sabok)

For any ideal $J$ on a countable set the following are equivalent:
(a) $J$ is represented on a Polish space;
(b) $J$ is dense and countably separated;
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For any ideal $J$ on a countable set the following are equivalent:
(a) $J$ is represented on a Polish space;
(b) $J$ is dense and countably separated;
(c) $J$ is represented on a compact space.
Definition

Given two ideals $J$, $K$ on $\omega$ we write $J \leq_{RB} K$ and say that $J$ is Rudin-Blass below $K$ if there is a finite-to-one $f : \omega \to \omega$ such that

$$a \in K \iff f^{-1}[a] \in J,$$

for every $a \subset \omega$. 

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The Rudin-Blass reduction

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$J$ and $K$ are Rudin-Blass equivalent if $J \leq_{RB} K$ and $K \leq_{RB} J$. 

Corollary (K.-Sabok)

The class of ideals represented on Polish spaces is invariant under Rudin-Blass equivalence.
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Corollary (K.-Sabok)

The class of ideals represented on Polish spaces is invariant under Rudin-Blass equivalence.
Descriptive complexity of ideals represented on Polish spaces

Theorem (K.-Sabok)

If $J$ is an analytic ideal represented on a Polish space, then it is $\Pi^0_3$-complete.

Corollary (K.-Sabok)

If $J$ is a coanalytic ideal represented on a Polish space, then it is either $\Pi^0_3$-complete or $\Pi^1_1$-complete.
Theorem (K.-Sabok)

If $J$ is an analytic ideal represented on a Polish space, then it is $\Pi_3^0$-complete.

Corollary (K.-Sabok)

If $J$ is a coanalytic ideal represented on a Polish space, then it is either $\Pi_3^0$-complete or $\Pi_1^1$-complete.
Thank you!