Representing ideals on Polish spaces

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Joint work with Marcin Sabok

Ideals represented on Polish spaces

Definition

Suppose that X is a Polish space and I is a σ-ideal on X containing all singletons. Given a dense countable set $D \subset X$ we define the ideal

$$J_I = \{a \subset D : cl(a) \in I\}.$$

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Given an ideal J *on a countable set* E *we say that* J *is represented on a Polish space if there are* X, I, D *as above and a bijection* $\rho : E \to D$ *such that* $J = \{a \subset E : \rho[a] \in J_I\}$.

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Examples

$NWD(\mathbb{Q}) = \{a \subset \mathbb{Q} \cap [0,1] : a \textit{ is nowhere dense}\}$
$NULL(\mathbb{Q}) = \{a \subset \mathbb{Q} \cap [0,1] : cl(a) \textit{ is of Lebesgue measure zero}\}$

Motivation

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For an ideal J on ω the equivalence relation E_J on 2^ω is given by $xE_Jy \Leftrightarrow x\triangle y \in J$.

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Let J be an ideal represented on a compact space.
(a) Suppose that E is an equivalence relation of a turbulent action. Every Borel homomorphism from E to E_J maps a comeager set to a single E_J-equivalence class.

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Theorem (Zapletal, 2012)

Let J be an ideal represented on a compact space.
(a) Suppose that E is an equivalence relation of a turbulent action. Every Borel homomorphism from E to E_J maps a comeager set to a single E_J-equivalence class.
(b) Suppose that J is represented by a $\mathbf{\Pi}^0_2$ σ-ideal of compact sets. Every Borel homomorphism from E_J to countable structures maps a comeager set to a single equivalence class.

Characterization of ideals represented on Polish spaces

Conjecture (Sabok-Zapletal)

For any ideal J on a countable set the following are equivalent:
(a) J is represented on a compact space;
(b) J is dense $\mathbf{\Pi}^0_3$ and weakly selective.

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Definition

We say that an ideal J on a countable set D is weakly selective if for any $a \notin J$ and any $f : a \to \omega$ there is $b \subset a$ with $b \notin J$ such that f restricted to b is either one-to-one or constant.

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We say that an ideal J on a countable set D is countably separated if there is a countable family $\{x_n : n \in \omega\}$ of subsets of D such that for any $a, b \subset D$ with $a \notin J$ and $b \in J$ there is $n \in \omega$ with $a \cap x_n \notin J$ and $b \cap x_n = \emptyset$.

Characterization of ideals represented on Polish spaces

Main Theorem (K.-Sabok)

For any ideal J on a countable set the following are equivalent:
(a) J is represented on a Polish space;
(b) J is dense and countably separated;

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For any ideal $\mathcal{J}$ on a countable set the following are equivalent:
(a) $\mathcal{J}$ is represented on a Polish space;
(b) $\mathcal{J}$ is dense and countably separated;
(c) $\mathcal{J}$ is represented on a compact space.

The Rudin-Blass reduction

Definition

Given two ideals J, K on ω we write $J \leq_{RB} K$ and say that J is Rudin-Blass below K if there is a finite-to-one $f : \omega \to \omega$ such that

$$a \in K \Leftrightarrow f^{-1}[a] \in J,$$

for every $a \subset \omega$.

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J and K are Rudin-Blass equivalent if $J \leq_{RB} K$ and $K \leq_{RB} J$.

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J and K are Rudin-Blass equivalent if $J \leq_{RB} K$ and $K \leq_{RB} J$.

Corollary (K.-Sabok)

The class of ideals represented on Polish spaces is invariant under Rudin-Blass equivalence.

Descriptive complexity of ideals represented on Polish spaces

Theorem (K.-Sabok)

If J is an analytic ideal represented on a Polish space, then it is $\mathbf{\Pi}^0_3$-complete.

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Corollary (K.-Sabok)

If J is a coanalytic ideal represented on a Polish space, then it is either $\mathbf{\Pi}^0_3$-complete or $\mathbf{\Pi}^1_1$-complete.

Thank you!