Cardinal Invariants of porous spaces

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Definition

Let $\langle X, d \rangle$ be a metric space. A subset $A \subseteq X$ is **strongly porous** if there exist a $p > 0$ such that for every $x \in X$ and every $r \in (0, \text{diam}X)$, there is $y \in X$ such that $B_{pr}(y) \subseteq B_r(x) \setminus A$.

Let's call $\text{SP}(X)$ the $\sigma$-ideal generated by strongly porous sets of $X$.

There are many concepts regarding porosity. One of them caught the attention of J. Brendle and R. Repický.
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Strongly porous sets
non(SP) vs add(N)
non(SP) sub n

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Theorem (J. Brendle, R. Repický)
\[
\text{add}(\text{UP}) = \omega_1, \quad \text{cof}(\text{UP}) = \mathfrak{c}, \quad \text{cov}(\text{UP}) \leq \text{cov}(\mathcal{N}), \quad \text{non}(\text{UP}) \geq \mathfrak{p}, \quad \text{non}(\text{UP}) \geq \text{add}(\mathcal{N})
\]

Theorem (M. Hrušák, O. Zindulka)

*It is consistent with ZFC that* \(\text{cov}(\text{SP}) > \text{cof}(\mathcal{N})\) *and that* \(\text{non}(\text{SP}) < \mathfrak{p}\)*

Our goal is to prove the consistency of \(\text{non}(\text{SP}) > \text{add}(\mathcal{N})\)*
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\begin{align*}
\text{add}(\text{UP}) &= \omega_1, \\
\text{cof}(\text{UP}) &= c, \\
\text{cov}(\text{UP}) &\leq \text{cov}(\mathcal{N}), \\
\text{non}(\text{UP}) &\geq p, \\
\text{non}(\text{UP}) &\geq \text{add}(\mathcal{N})
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Theorem (M. Hrušák, O. Zindulka)

*It is consistent with ZFC that cov(SP) > cof(\mathcal{N}) and that non(SP) < p*

Our goal is to prove the consistency of non(SP) > add(\mathcal{N})
¿What can we say about the cardinal invariants of $\text{SP} (\mathbb{R})$?

**Theorem**

- $\text{add} (\text{SP} (\mathbb{R})) = \text{add} (\text{SP} (2^\omega))$.
- $\text{cov} (\text{SP} (\mathbb{R})) = \text{cov} (\text{SP} (2^\omega))$.
- $\text{non} (\text{SP} (\mathbb{R})) = \text{non} (\text{SP} (2^\omega))$.
- $\text{cof} (\text{SP} (\mathbb{R})) = \text{cof} (\text{SP} (2^\omega))$. 
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- $\text{cof}(\text{SP}(\mathbb{R})) = \text{cof}(\text{SP}(2^{\omega}))$. 
**Lemma**

A subset $A \subseteq 2^\omega$ is strongly porous iff there is a $n \in \omega$ such that for every $p \in 2^{<\omega}$ there is $q \in 2^{<\omega}$ such that $p \subseteq q$, $|q| = |p| + n$ and $A \cap \langle q \rangle = \emptyset$.

**Definition**

Let $A \subseteq 2^\omega$. Let's say that $A$ is a strongly porous set of $n$ degree if for every $p \in 2^{<\omega}$ there is $q \in 2^{<\omega}$ such that $p \subseteq q$, $|q| = |p| + n$ and $A \cap \langle q \rangle = \emptyset$.

Therefore $A \subseteq 2^\omega$ is strongly porous iff there exists $n$ such that $A$ is strongly porous of $n$ degree. Let's call $\text{SP}_n$ the $\sigma$-ideal generated by strongly porous subsets of $n$ degree.
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A forcing \( \mathbb{P} \) strongly preserves non(\( \text{SP}_n \)) if for every \( \dot{X} \), a \( \mathbb{P} \) name for a porous set of \( n \) degree, there is \( Y \in \text{SP}_n \) such that for every \( x \in 2^\omega \), if \( x \notin Y \), then \( \Vdash_\mathbb{P} \text{"} x \notin \dot{X} \" \).

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If \( \mathbb{P} \) strongly preserves non(\( \text{SP}_n \)), then \( V[G] \models 2^\omega \cap V \notin \text{SP}_n \).
Definition

A forcing $\mathbb{P}$ strongly preserves non($\text{SP}_n$) if for every $\dot{X}$, a $\mathbb{P}$ name for a porous set of $n$ degree, there is $Y \in \text{SP}_n$ such that for every $x \in 2^\omega$, if $x \notin Y$, then $\Vdash_{\mathbb{P}} "x \notin \dot{X}"$.

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If $\mathbb{P}$ strongly preserves non($\text{SP}_n$), then $V[G] \models 2^\omega \cap V \notin \text{SP}_n$. 
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Lemma
Let $\mathbb{P}$ be a $\sigma\left(2^n\right)$-linked forcing, then $\mathbb{P}$ strongly preserves non(SP$_n$).

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Finite iteration of c.c.c. forcings which strongly preserves non(SP$_n$), strongly preserves non(SP$_n$).
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Let
\[ \mathbb{A} = \{B \in \text{Borel}(2^\omega) : \mu(B) > \frac{1}{2}\} \]
and lets say that $A \leq B$ iff $A \subseteq B$. This is called the amoeba forcing.

**Lemma**

For every $n \in \omega$, $\mathbb{A}$ is a $\sigma$ $n$-linked forcing.

Therefore $\mathbb{A}$ preserves non($\text{SP}_n$) for every $n \in \omega$.

**Lemma**

If $G$ is a generic filter over $\mathbb{A}$, then $V[G] \models \mu(\bigcup(\mathcal{N} \cap V)) = 0$. 
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**Theorem**

*It is consistent with ZFC that $\text{non}(\text{SP}) < \text{add}(\mathcal{N})$.***

Start with a model of CH and consider a finite support iteration of length $\omega_2$ of amoeba forcing. If we have an uncountable family $\mathcal{N}$ of null sets, then this family is encoded in a middle step of the iteration. Then, by the previous lemma, the union of this family is a null set in the next step of the iteration. On the other hand, as this forcing strongly preserves $\text{non}(\text{SP})$, $\text{non}(\text{SP}) = \omega_1$. Wait! there’s more.
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Wait! there’s more.
What can we say about the cardinal non(\(\text{SP}_n\))? Let \(\mathbb{P}\) be the following forcing

\[
\mathbb{P}_n = \{ \langle s, F \rangle : \quad \begin{align*}
    (a) & \quad s; 2^{<\omega} \to 2^n, \\
    (b) & \quad |s| < \omega, \\
    (c) & \quad F \in [2^\omega]^{<\omega}, \\
    (d) & \quad \text{for every } \sigma \in \text{dom}(s), \quad F \cap \langle \sigma \smallfrown s(\sigma) \rangle = \emptyset,
\end{align*}
\]

we say that \(\langle s, F \rangle \leq \langle s', F' \rangle\) iff \(s' \subseteq s\) and \(F' \subseteq F\).
Lemma

\( \mathbb{P}_n \) is a \( \sigma (2^n - 1) \)-linked forcing.

Lemma

Let \( G \) be a \( \mathbb{P}_n \) generic filter over a ground model \( M \). Then
\[ V[G] \models 2^\omega \cap V \in \text{SP}_n. \]

(\( \mathbb{P}_n \) can’t be a \( \sigma (2^n) \)-linked forcing.)

Theorem

For every \( n \in \omega \) and for every \( k < 2^n \), \( m_{\sigma k} \)-linked \( \leq \) \( \text{non}(\text{SP}_n) \).
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$\mathbb{P}_n$ is a $\sigma\,(2^n - 1)$-linked forcing.

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*It is consistent with ZFC that \( \text{non}(\text{SP}_n) < \text{non}(\text{SP}_{n+1}) \).*

Start with a ground model of ZFC + CH. Consider a finite support iteration of length \( \omega_2 \) of the forcing \( P_{n+1} \). As all of these forcings strongly preserve \( \text{non}(\text{SP}_1) \), then \( \text{non}(\text{SP}_n) = \omega_1 \). On the other hand, a reflection argument shows that \( \text{non}(\text{SP}_{n+1}) \geq \omega_2 \).
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