Cardinal Invariants of porous spaces

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Definition

Let $\langle X, d\rangle$ be a metric space. A subset $A \subseteq X$ is *strongly porous* if there exist a $p > 0$ such that for every $x \in X$ and every $r \in (0, \mathrm{diam} X)$, there is $y \in X$ such that $B_{pr}(y) \subseteq B_r(x) \setminus A$.

Lets call $\mathbf{SP}(X)$ the σ-ideal generated by strongly porous sets of X.

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Theorem (J. Brendle, R. Repický)

$\mathrm{add}(\mathbf{UP}) = \omega_1$, $\mathrm{cof}(\mathbf{UP}) = \mathfrak{c}$, $\mathrm{cov}(\mathbf{UP}) \leq \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathbf{UP}) \geq \mathfrak{p}$, $\mathrm{non}(\mathbf{UP}) \geq \mathrm{add}(\mathcal{N})$

Theorem (M. Hrušák, O. Zindulka)

It is consistent with ZFC that $\mathrm{cov}(\mathbf{SP}) > \mathrm{cof}(\mathcal{N})$ *and that* $\mathrm{non}(\mathbf{SP}) < \mathfrak{p}$

Our goal is to prove the consistency of $\mathrm{non}(\mathbf{SP}) > \mathrm{add}(\mathcal{N})$

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add($\mathbf{UP}$) $= \omega_1$, cof($\mathbf{UP}$) $= \mathfrak{c}$, cov($\mathbf{UP}$) $\leq$ cov($\mathcal{N}$), non($\mathbf{UP}$) $\geq \mathfrak{p}$, non($\mathbf{UP}$) $\geq$ add($\mathcal{N}$)

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¿What can we say about the cardinal invariants of $\mathbf{SP}(\mathbb{R})$?

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Lemma

A subset $A \subseteq 2^\omega$ is strongly porous iff there is a $n \in \omega$ such that for every $p \in 2^{<\omega}$ there is $q \in 2^{<\omega}$ such that $p \subseteq q$, $|q| = |p| + n$ and $A \cap \langle q \rangle = \emptyset$.

Definition

Let $A \subseteq 2^\omega$. Lets say that A is a *strongly porous set of* n degree if for every $p \in 2^{<\omega}$ there is $q \in 2^{<\omega}$ such that $p \subseteq q$, $|q| = |p| + n$ and $A \cap \langle q \rangle = \emptyset$.

Therefore $A \subseteq 2^\omega$ is strongly porous iff there exists n such that A is strongly porous of n degree.
Lets call $\mathbf{SP}_n$ the σ-ideal generated by strongly porous subsets of n degree.

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A forcing $\mathbb{P}$ strongly preserves non($\mathbf{SP}_n$) if for every $\dot{X}$, a $\mathbb{P}$ name for a porous set of n degree, there is $Y \in \mathbf{SP}_n$ such that for every $x \in 2^\omega$, if $x \notin Y$, then $\Vdash_{\mathbb{P}}$ "$x \notin \dot{X}$".

Lemma

If $\mathbb{P}$ strongly preserves non($\mathbf{SP}_n$)*, then $V[G] \models 2^\omega \cap V \notin \mathbf{SP}_n$.*

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Let

$$\mathbb{A} = \{B \in Borel(2^\omega) : \mu(B) > \frac{1}{2}\}$$

and lets say that $A \leq B$ iff $A \subseteq B$. This is called the amoeba forcing.

Lemma

For every $n \in \omega$, $\mathbb{A}$ is a σ n-linked forcing.

Therefore $\mathbb{A}$ preserves non($\mathbf{SP}_n$) for every $n \in \omega$.

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If G is a generic filter over $\mathbb{A}$, then $V[G] \models \mu(\bigcup(\mathcal{N} \cap V)) = 0$.

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Theorem

It is consistent with ZFC that $\text{non}(\mathbf{SP}) < \text{add}(\mathcal{N})$.

Start with a model of CH and consider a finite suport iteration of lenght ω_2 of amoeba forcing. If we have an uncountable family $\mathcal{N}$ of null sets, then this family is encoded in a middle step of the iteration. Then, by the previous lemma, the union of this family is a null set in the next step of the iteration. On the other hand, as this forcing strongly preserves $\text{non}(\mathbf{SP})$, $\text{non}(\mathbf{SP}) = \omega_1$. Wait! there's more.

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Wait! there's more.

What can we say about the cardinal non($\mathbf{SP}_n$)?
Let $\mathbb{P}$ be the following forcing

$$\mathbb{P}_n = \{\langle s, F\rangle : \quad \begin{array}{l} \textbf{(a)}\ s; 2^{<\omega} \to 2^n, \\ \textbf{(b)}\ |s| < \omega, \\ \textbf{(c)}\ F \in [2^\omega]^{<\omega}, \\ \textbf{(d)}\ \text{for every } \sigma \in \mathrm{dom}(s),\ F \cap \langle \sigma^\frown s(\sigma)\rangle = \emptyset, \end{array}$$

we say that $\langle s, F\rangle \leq \langle s', F'\rangle$ iff $s' \subseteq s$ and $F' \subseteq F$.

Lemma

$\mathbb{P}_n$ is a σ $(2^n - 1)$-linked forcing.

Lemma

Let G be a $\mathbb{P}_n$ generic filter over a ground model M. Then $V[G] \models 2^\omega \cap V \in \mathbf{SP}_n$.

($\mathbb{P}_n$ can't be a σ (2^n)-linked forcing.)

Theorem

For every $n \in \omega$ and for every $k < 2^n$, $\mathfrak{m}_{\sigma\ k\text{-linked}} \leq \mathrm{non}(\mathbf{SP}_n)$.

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It is consistent with ZFC that $\text{non}(\mathbf{SP}_n) < \text{non}(\mathbf{SP}_{n+1})$.

Start with a ground model of ZFC + CH. Consider a finite support iteration of lenght ω_2 of the forcing $\mathbb{P}_{n+1}$. As all of these forcings strongly preserve $\text{non}(\mathbf{SP}_1)$, then $\text{non}(\mathbf{SP}_n) = \omega_1$. On the other hand, a reflection argument shows that $\text{non}(\mathbf{SP}_{n+1}) \geq \omega_2$.

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Here are some questions for you.

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