Unique homogeneity, III

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Arhangelskii-vM Theorems that are not so surprising.

Theorem

Let X be UH such that $|X| > 2$. Then X is connected.

Let $\mathcal{C}$ denote the collection of all clopen subsets of X. For every $x \in X$ we let

$$Q_x = \bigcap\{C : x \ni C \in \mathcal{C}\}$$

denote the *quasi-component* of x. Assume that X is not connected. Then there exist $a, b \in X$ such that $Q_a \cap Q_b = \emptyset$. Let U and V be clopen subsets of X such that $a \in U \not\ni b$, $b \in V \not\ni a$. Let $h\colon X \to X$ be the unique homeomorphism taking a onto b. Let $W = U \cap h^{-1}(V)$. Then $a \in Q_a \subseteq W$, $b \in Q_b \subseteq h(W)$ and $h(Q_a) = Q_b$. Moreover, $W \cap h(W) = \emptyset$.

CASE 1: $W = Q_a$.

Then Q_a is open. Since X is homogeneous and every homeomorphism of X permutes the quasi-components of X, it follows that Q_a is homogeneous. Assume that Q_a contains more than one point. It therefore has a nontrivial homeomorphism that can be extended to a homeomorphism of X by requiring it to be the identity outside Q_a. But this homeomorphism fixes b, hence must be the identity. Contradiction. Hence Q_a is a singleton, hence X is discrete and hence not UH since $|X| \geq 2$.

CASE 2: $W \neq Q_a$.

Then pick an element $c \in W \setminus Q_a$. There is a clopen subset W_1 of X such that $Q_a \subseteq W_1 \subseteq W \setminus \{c\}$. Then $h(W_1) \subseteq h(W)$ and hence misses W. We can now interchange W_1 and $h(W_1)$ and extend this homeomorphism with the identity on the complement of $W_1 \cup h(W_1)$. Since this homeomorphism fixes c and is not the identity, we reached a contradiction.

Theorem

No infinite subspace of an ordered space is UH.

There are several proofs of this result, the one below is elementary.

Let X be an infinite UH subspace of an ordered space. Then X is connected by the previous theorem, hence X is ordered. It clearly has no smallest or largest element by homogeneity.

Fix $e \in X$ for some time.

CLAIM 1: If $f, g \in H(X)$ and $f(e) < g(e)$, then $f(x) < g(x)$ for *every* $x \in X$.

Let $M = \{x \in X : f(x) < g(x)\}$, $L = \{x \in X : g(x) < f(x)\}$. Then both M and L are open, $M \cap L = \emptyset$ and $e \in M$. Since

$f(e) < g(e)$ it follows that $f(x) \neq g(x)$ for every x, hence $X = M \cup L$. Since $M \neq \emptyset$ and X is connected, $M = X$.

CLAIM 2: If $g \in H(X)$ and there exists $x \in X$ such that $e < g(x)$, then $x < g(x)$ for every $x \in X$.

This is just Claim 1 with f taken to be the identity on X. Observe that Claims 1 and 2 were proved for arbitrary $e \in X$.

CLAIM 3: Every homeomorphism on X is strictly increasing.

Take an arbitrary $f \in H(X) \setminus \{\mathrm{id}_X\}$.

CASE 1: There exists $e \in X$ such that $e < f(e)$.

Then $x < f(x)$ for every $x \in X$ by Claim 2. Assume that there exist $x, y \in X$ such that $x < y$ and $f(y) < f(x)$. Then $Ef([x, \rightarrow)] = (\leftarrow, f(x)]$. Now pick $z \in X$ such that $f(x) < z$. Then on the one hand $z < f(z)$ while on the other hand

$x < f(x) < z$, hence $f(z) < f(x) < z$. This is a contradiction.

CASE 2: There exists $e \in X$ such that $f(e) < e$.

Now replace f by f^{-1} and conclude by Claim 1 that f^{-1} is strictly increasing, hence f is strictly increasing as well.

Again, for $a, b \in X$ let f_b^a be the unique homeomorphism of X that takes a onto b.

Fix $e \in X$. For every $x \in X$, put $i(x) = (f_x^e)^{-1}(e)$. Observe that $i(e) = (f_e^e)^{-1}(e) = e$.

CLAIM 4: i is a bijection of X reversing the order.

Pick $a \in X$ such that $e \leq a$. Observe that $e \leq f_a^e(e) = a$. Hence $x \leq f_a^e(x)$ holds for all $x \in X$ (Claim 2). This means that

$$i(a) \leq f_a^e(i(a)) = f_a^e\big((f_a^e)^{-1}(e)\big) = e.$$

Similarly, $i(b) \geq e$ for every $b \leq e$.

Now take $a, b \in X$ such that $a < b$. If $a \leq e < b$, then by what we just proved, $i(b) < e \leq i(a)$.

CASE 1: $e < a < b$.

SUBCASE 1: $i(b) = i(a)$.

Then

$$f_a^e(i(b)) = f_a^e(i(a)) = e = f_b^e(i(b)).$$

Hence $f_a^e = f_b^e$, i.e., $a = f_a^e(e) = f_b^e(e) = b$ which is a contradiction.

SUBCASE 2: $i(a) < i(b)$.

Then $f_b^e(i(a)) < f_b^e(i(b)) = e$ since f_b^e is strictly increasing (Claim 3). Now since $a = f_a^e(e) < f_b^e(e) = b$ we have $f_a^e(x) < f_b^e(x)$ for every $x \in X$ (Claim 1). From this we conclude that

$$e = f_a^e(i(a)) < f_b^e(i(a))$$

which contradicts the above.

So indeed $i(b) < i(a)$.

CASE 2: $a < b < e$.

Similar reasoning.

Hence i is an order reversing homeomorphism with a fixed point. Contradiction.

Question

1. Does there exist a Polish UH space? *It cannot be locally compact*
2. Is there a compact uniquely homogeneous space? *It cannot be metrizable and it cannot be a topological group (W. Rudin)*
3. Do there exist uniquely homogeneous spaces X of arbitrarily large weight? *Yes for all* $w(X) \leq 2^{\mathfrak{c}}$