Tukey reduction

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Outline of Topics

1. Tukey reduction and basic orders
2. Ideals
3. Structure of Tukey reduction among ideals
All the unattributed results are due to Todorcevic and myself.
Tukey reduction and basic orders
Tukey reduction
A directed order \((D, \leq)\) is a partial order such that for each \(x, y \in D\) there is \(z \in D\) with \(x, y \leq z\).

A set \(A \subseteq D\) is called **bounded** if there is \(x \in D\) such that \(y \leq x\) for each \(y \in A\).

A set \(A \subseteq D\) is **cofinal** if for each \(x \in D\) there \(y \in A\) with \(x \leq y\).
$D$ and $E$ directed orders.

A function $f : D \to E$ is called **Tukey** if preimages under $f$ of sets bounded in $E$ are bounded in $D$. We write

$$D \leq_T E$$

if there is a Tukey function from $D$ to $E$. 

If $D$ and $E$ are Tukey reducible to each other, we say that they are **Tukey equivalent**, and we write

$$D \equiv_T E.$$ 

**Theorem (Tukey)**

Let $D$ and $E$ be directed order. Then $D \equiv_T E$ if and only if $D$ and $E$ can be embedded as cofinal subsets of a directed order.
Dual point of view: A function $g : E \to D$ is convergent if images under $g$ of sets cofinal in $E$ are cofinal in $D$.

For two directed orders $D$ and $E$, there is a Tukey function from $D$ to $E$ if and only if there is a convergent function from $E$ to $D$. 
Examples.

\[ \mathbb{N} \prec_T \mathbb{N}^\mathbb{N} \]

\[ \mathbb{N} \not\leq_T \omega_1, \ \omega_1 \not\leq_T \mathbb{N} \]
Connection with cardinal invariants.

\[ \text{add}(D) = \text{minimal cardinality of an unbounded subset of } D \]

\[ \text{cof}(D) = \text{minimal cardinality of a cofinal subset of } D. \]

\[ D \leq_T E \implies \text{add}(E) \leq \text{add}(D) \text{ and } \text{cof}(D) \leq \text{cof}(E). \]
Basic orders
A directed order $D$ is called \textbf{basic} if

- $D$ is a separable metric space;
- each two elements of $D$ have the least upper bound and the operation of taking the least upper bound is a continuous function from $D \times D$ to $D$;
- each bounded sequence has a convergent subsequence;
- each convergent sequence has a bounded subsequence.
Examples of basic orders.

1. $\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$

2. NWD all *closed nowhere dense* subsets of $2^\mathbb{N}$ taken with inclusion as the directed order relation. View NWD as a subset of the compact space $\mathcal{K}(2^\mathbb{N})$ with the Vietoris topology.

3. $\ell_1$ all subsets $\pi$ of $\mathbb{N}$ with

$$\sum_{n \in \pi} \frac{1}{n + 1} < \infty$$

taken with inclusion as the directed order relation. View $\ell_1$ with the topology given by the following metric

$$d(\pi, \phi) = \sum_{n \in \pi \Delta \phi} \frac{1}{n + 1}.$$
A separable metric is called **analytic** if it is a continuous image of a Polish space. For example, all Borel subsets of Polish spaces are analytic.

Basic orders whose underlying topology is analytic are called **analytic basic orders**.

All the examples above are analytic basic orders.
Analytic basic orders form an initial class of basic orders.

**Theorem**

Let $D$ and $E$ be basic orders. If $E$ is analytic and $D \leq_T E$, then $D$ is analytic.
**Theorem**

Let $D$ be a basic order. If the topology on $D$ is analytic, then it is Polish.

**Theorem**

Let $D$ and $E$ be analytic basic orders. If $D \leq_T E$, then there exist a Tukey function from $D$ to $E$ that is measurable with respect to the $\sigma$-algebra generated by analytic sets.
The interesting analytic basic orders are the non-locally compact ones: $\mathbb{N}^\mathbb{N}$, NWD, $\ell_1$; not $\mathbb{N}$.

**Proposition**

Let $D$ be an analytic non-locally compact basic order. Then $\mathbb{N}^\mathbb{N} \leq_T D$. 
Back to cardinal invariants.

\( \text{MGR} = \) all meager subsets of \( 2^\mathbb{N} \) taken with inclusion

\( \text{NULL} = \) all Lebesgue measure zero subsets of \([0, 1]\) taken with inclusion

These are directed orders that are not basic orders. Cardinal invariants

\[
\text{add/cof}(\text{MGR}) \text{ and } \text{add/cof}(\text{NULL})
\]

are of interest.
A set is $\sigma$-bounded if it is a countable union of bounded sets.

$D, E$ directed orders

$$D \leq_T E$$

if there is a function $D \rightarrow E$ such that preimages of $\sigma$-bounded sets are $\sigma$-bounded.

$$D \equiv_T E$$

if both $D \leq_T E$ and $E \leq_T D$.

Note: $D \leq_T E$ implies $D \leq_T E$. 
add^\omega(D) = \text{minimal cardinality of a non-}\sigma\text{-bounded subset of } D.

D \leq_T E \implies \text{add}^\omega(E) \leq \text{add}^\omega(D), \text{cof}(D) \leq \max(\omega, \text{cof}(E)).
Theorem (Bartoszyński, Raissonier–Stern, Fremlin)

\[ \text{MGR} \equiv_T \omega \text{NWD and NULL} \equiv_T \omega \ell_1. \]

So

\[ \text{add}(\text{MGR}) = \text{add}^\omega(\text{NWD}), \quad \text{cof}(\text{MGR}) = \text{cof}(\text{NWD}) \]

\[ \text{add}(\text{NULL}) = \text{add}^\omega(\ell_1), \quad \text{cof}(\text{NULL}) = \text{cof}(\ell_1). \]

So \( \text{NWD} \leq_T \ell_1 \) would give

\[ \text{add}(\text{NULL}) \leq \text{add}(\text{MGR}) \quad \text{and} \quad \text{cof}(\text{MGR}) \leq \text{cof}(\text{NULL}). \]
Ideals
The main class of examples of basic orders are ideals taken with inclusion. The world is divided into a **compact part** \((\sigma\text{-ideals, category leaf})\) and a **discrete part** \((P\text{-ideals, measure leaf})\).
$\sigma$-ideals
Ideals

Let $X$ be a compact metric space.

The space $\mathcal{K}(X) = \{\text{all compact subsets of } X \text{ with the Vietoris topology}\}$ is a compact metric space.

A set $\mathcal{I} \subseteq \mathcal{K}(X)$ is a $\sigma$-ideal of compact sets if it is closed under taking compact subsets and countable compact unions.

A $\sigma$-ideal of compact sets with inclusion and the topology inherited from $\mathcal{K}(X)$ is a basic order.
Kečrís–Louveau–Woodin: a \( \sigma \)-ideal \( \mathcal{I} \) of compact sets is locally compact if and only if \( \mathcal{I} = \mathcal{K}(U) \) for some open set \( U \subseteq X \).

**Convention**: a **\( \sigma \)-ideal** is an analytic, non-locally compact \( \sigma \)-ideal of compact subsets of a compact metric space.
A $\sigma$-ideal $\mathcal{I}$ has property (\textasteriskcentered) if for each sequence $(K_n)$ of sets in $\mathcal{I}$ there is a $G_\delta$ subset $G$ of $X$ such that $\bigcup_n K_n \subseteq G$ and all compact subsets of $G$ are in $\mathcal{I}$.

Fact of nature: all naturally occurring $\sigma$-ideals have (\textasteriskcentered).
Examples.

1. $\mathbb{N}^\mathbb{N}$ is Tukey equivalent to the $\sigma$-ideal with $\{\star\} \mathcal{K}(\mathbb{R} \setminus \mathbb{Q})$.

2. NWD is a $\sigma$-ideal with $\{\star\}$.

3. Mátrai: there is a $\sigma$-ideal without $\{\star\}$.

I found the following example $\mathcal{I}_0$. 
Consider $\bar{s} = (s_0, s_1, \ldots)$ infinite or finite with an even number of entries, each $s_i$ is a function from a non-empty finite subset of $\mathbb{N}$ to 2, for each $i$, $\text{dom}(s_i) < \text{dom}(s_{i+1})$.

Let $\mathcal{R}$ be the set of all such sequences.

For $\bar{s} \in \mathcal{R}$, define

$$[\bar{s}] = \{ x \in 2^\mathbb{N} : s_{2i} \subseteq x \text{ or } s_{2i+1} \subseteq x \text{ for each } i \}.$$

Define

$$\mathcal{I}_0 = \{ K \in \mathcal{K}(2^\mathbb{N}) : K \cap [\bar{s}] \text{ is nowhere dense in } [\bar{s}] \text{ for each } \bar{s} \in \mathcal{R} \}.$$

$\mathcal{I}_0$ is a $\sigma$-ideal without ($\ast$).
P-ideals
A set $I \subseteq \mathcal{P}(\mathbb{N})$ is a \textbf{P-ideal of subsets of $\mathbb{N}$} if it is closed under taking finite unions and subsets and for each sequence $x_n \in I$, $n \in \mathbb{N}$, there is $x \in I$ such that $x_n \setminus x$ is finite for each $n$.

View $I$ as a subspace of the compact metric space $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$. 
$S.$: $I$ is an analytic $P$-ideal of subsets of $\mathbb{N}$ if and only if there exists a lower semicontinuous submeasure $\phi: \mathcal{P}(\mathbb{N}) \to [0, \infty]$ such that

$$I = \text{Exh}(\phi) = \{x \in \mathcal{P}(\mathbb{N}) : \lim_{n} \phi(x \setminus n) = 0\}.$$  

Such an $I$ becomes a Polish space with the submeasure topology given by the metric

$$d_\phi(x, y) = \phi(x \triangle y).$$
An analytic P-ideal of subsets of $\mathbb{N}$ taken with inclusion and with its submeasure topology is an analytic basic order.

S.–Todorcevic: if an ideal $I \subseteq \mathcal{P}(\mathbb{N})$ taken with inclusion and with a topology $\tau$ containing the topology inherited from $\mathcal{P}(\mathbb{N})$ is an analytic basic order, then $I$ is an analytic P-ideal and $\tau$ is the submeasure topology.
S.: an analytic P-ideal $I$ of subsets of $\mathbb{N}$ is locally compact with its submeasure topology if and only if $I = \{ x \in \mathcal{P}(\mathbb{N}) : x \cap a \text{ is finite} \}$ for some $a \subseteq \mathbb{N}$.

**Convention:** a **P-ideal** is an analytic, non-locally compact P-ideal of subsets of $\mathbb{N}$.
$I = \text{Exh}(\phi)$ a P-ideal for a lower semicontinuous submeasure $\phi$.

$I$ density-like if for each $\epsilon > 0$ there is $\delta > 0$ such that for each sequence $(x_n)$ of sets in $I$ with $\phi(x_n) < \delta$ there are $n_0 < n_1 < n_2 < \cdots$ with

$$\phi\left(\bigcup_k x_{n_k}\right) < \epsilon.$$
Examples.

1. $\mathbb{N}^\mathbb{N}$ is Tukey equivalent to the density-like $\mathcal{P}$-ideal

$$
\emptyset \times \text{Fin} = \{ x \in \mathcal{P}(\mathbb{N} \times \mathbb{N}) : \forall m \{ n : (m, n) \in x \} \text{ is finite} \}.
$$

2. The ideal

$$
\mathcal{Z}_0 = \{ x \in \mathcal{P}(\mathbb{N}) : \lim_n \frac{|x \cap (n+1)|}{n+1} = 0 \}
$$

is a density-like $\mathcal{P}$-ideal.

A lower semicontinuous submeasure for $\mathcal{Z}_0$:

$$
\phi_0(x) = \sup_n \frac{|x \cap (n+1)|}{n+1}.
$$
3. $\ell_1$ is a $P$-ideal that is not density-like.

A lower semicontinuous submeasure for $\ell_1$:

$$\phi_1(x) = \sum_{n \in x} \frac{1}{n + 1}.$$
Structure of Tukey reduction among ideals
Within classes
Theorem (Louveau–Veličković, Todorcevic)

\[ \ell_1 \text{ is Tukey largest among } P\text{-ideals}. \]

Is there a Tukey largest \( \sigma \)-ideal?
Theorem (Louveau–Veličković)

There is an embedding of the partial order $\mathcal{P}(\mathbb{N})/\text{Fin}$ with almost inclusion into the class of $P$-ideals with Tukey reduction.

Is the analogous result true for $\sigma$-ideals?
Theorem (S.)

NWD is Tukey largest among $\sigma$-ideals with $(\ast)$.

Is there a Tukey largest density-like P-ideal?
Across classes
There are essentially no Tukey reduction from the P-ideals to $\sigma$-ideals.

**Theorem**

If $I$ is a P-ideal, $\mathcal{I}$ a $\sigma$-ideal, and $I \leq_T \mathcal{I}$, then $I$ is isomorphic to $\emptyset \times \text{Fin}$, so $I \equiv_T \mathbb{N}^\mathbb{N}$.

So if a P-ideal is Tukey equivalent to a $\sigma$-ideal, then they are both Tukey equivalent to the smallest analytic non-locally compact basic order $\mathbb{N}^\mathbb{N}$. 
Among examples
Among the concrete examples defined above,

\[ \mathbb{N}^\mathbb{N}, \text{NWD}, \mathcal{I}_0, \mathcal{Z}_0, \ell_1, \]

the structure of Tukey reduction is completely known.
Theorem

(i) (Isbell, Fremlin, Louveau–Veličković)

\[ \mathbb{N}^\mathbb{N} <_T \mathcal{Z}_0 <_T \ell_1 \]

(ii) (Fremlin, Moore–Solecki)

\[ \mathbb{N}^\mathbb{N} <_T \text{NWD} <_T \mathcal{I}_0 \]
Theorem

(i) (Bartoszyński, Raissonier–Stern, Fremlin) $\text{NWD} \prec_T \ell_1$

(ii) (Mátrai, Solecki–Todorcevic) $\text{NWD} \not\leq_T \mathcal{Z}_0$

(iii) (Mátrai) $\mathcal{I}_0 \not\leq_T \ell_1$

From (i) we get

$$\text{add}(\text{NULL}) \leq \text{add}(\text{MGR}) \text{ and } \text{cof}(\text{MGR}) \leq \text{cof}(\text{NULL}).$$
Shadow of NWD
Recall: $\text{NWD} \leq_T \ell_1$ and $\text{NWD} \not\leq_T \mathbb{Z}_0$.

**Theorem**

Let $I$ be a density-like $P$-ideal. Then $\text{NWD} \not\leq_T I$.

Characterize those $P$-ideals $I$ for which $\text{NWD} \leq_T I$. 
Extracting an ordinal out of a P-ideal

\( \mathcal{I} \) a P-ideal

\( \mathcal{I} = \text{Exh}(\phi) \), for a lower semicontinuous submeasure \( \phi \)

Given a sequence \((x_n)\) of sets in \( \mathcal{I} \) and \( \epsilon > 0 \), the set

\[
\{ b \subseteq \mathbb{N} : \phi(\bigcup_{n \in b} x_n) \leq \epsilon \} \subseteq 2^\mathbb{N}
\]

is compact.

Let \textbf{height} of this set be \( \omega_1 \) if it contains an infinite set.

If it consists of finite sets only, let its \textbf{height} be \( \alpha \), where \( \alpha \) is such that its Cantor–Bendixson rank is \( \alpha + 1 \).
Let $\epsilon, \delta > 0$ and $\alpha \in \omega_1$ be given.

$P_{\epsilon,\delta}(\alpha)$ holds if for every sequence $(x_n)$ of sets in $I$ with $\phi(x_n) < \delta$

$$\text{height} \left( \{ b : \phi(\bigcup_{n \in b} x_n) \leq \epsilon \} \right) \geq \alpha.$$
Let

$$ht(I) = \min \{ \alpha \in \omega_1 : \exists \epsilon > 0 \forall \delta > 0 \ P_{\epsilon, \delta}(\alpha) \text{ fails} \}$$

if the set on the right hand side is non-empty, and let

$$ht(I) = \omega_1$$

otherwise.
Proposition

Let $I$ be a $P$-ideal. Then

(i) $\text{ht}(I)$ does not depend on the choice of submeasure $\phi$ with $I = \text{Exh}(\phi)$;

(ii) $\text{ht}(I) = \omega_1$ or $\text{ht}(I) = \omega^{\omega^\alpha}$ for some $\alpha < \omega_1$. 
A characterization of P-ideals with the largest and smallest values of height

Theorem

Let $I$ be a P-ideal. Then

(i) $ht(I) = \omega_1$ if and only if $I$ is density-like;

(ii) $ht(I) = \omega$ if and only if $I \equiv_T \ell_1$.

Height is an invariant of Tukey reduction.

Theorem

Let $I, J$ be P-ideals. If $I \leq_T J$, then $ht(J) \leq ht(I)$.
Is there an ordinal $\alpha$ such that

$$NWD \leq_T I \text{ if and only if } \text{ht}(I) \leq \alpha?$$